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MENTAL AND SOCIAL MEASUREMENTS

AN INTRODUCTION
TO THE
THEORY OF MENTAL AND SOCIAL MEASUREMENTS

BY

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PREFACE TO FIRST EDITION

Experience has sufficiently shown that the facts of human nature can be made the material for quantitative science. The direct transfer of methods originating in the physical sciences or in commerical arithmetic to sciences dealing with the complex and variable facts of human life has, however, resulted in crude and often fallacious measurements. Moreover, it has been difficult to teach students to estimate quantitative evidence properly or to obtain and use it wisely, because the books to which one could refer them were too abstract mathematically or too specialized, and omitted altogether much of the knowledge about mental measurements most needed by the majority of university students.

It is the aim of this book to introduce students to the theory of mental measurements and to provide them with such knowledge and practice as may assist them to follow critically quantitative evidence and argument and to make their own researches exact and logical. Only the most general principles are outlined, the special methods appropriate to each of the mental sciences being better left for separate treatment. If the general problems of mental measurement are realized and the methods at hand for dealing with variable quantities are mastered, the student will find no difficulty in acquiring the special information and technique involved in the quantitative aspect of his special science. The author has had in mind the needs of students of economics, sociology and education, possibly even more than those of students of psychology, pure and simple. Indeed, a great part of the discussion is relevant to the problems of anthropometry and vital statistics. The book may, with certain limitations, be used as an introduction to the theory of measurement of all variable phenomena.

Teachers College,
Columbia University, 1904.

PREFACE TO SECOND EDITION

Since the first edition of this book appeared the literature relating to methods of measuring mental and social facts has been enriched by a number of general accounts of such methods and by many reports of investigations in which they have been used. In particular, Brown's *The Essentials of Mental Measurement,* Yule's *Introduction to the Theory of Statistics* and Whipple's *Manual of Mental and Physical Tests,* make available for the English reader the same facts (and many more) as were outlined in this book.

I had hoped, consequently, that this book, having played a part in stimulating intelligent quantitative work in the mental and social sciences, would suffer a natural death. It is the case, however, that for the great majority of students of psychology, sociology and education, the abstract mathematical treatment, characteristic of the first two books mentioned, is out of question. In fact, an elementary introduction to the theory of mental measurements, treating the simpler general problems in the logic of quantitative thinking, is needed now more than ever. The increased use of modern methods in measuring conditions, differences, changes, and relations, including correlations or resemblances, requires that even those students of the mental and social sciences who will themselves never undertake original quantitative work should be able to interpret such results as the modern methods present. So this book is reissued.

It has been revised, and the greater part of it entirely rewritten, to fit the new conditions—that is, to introduce the students to the literature on mental and social measurements which is now available—and also to fit the abilities and needs of students. In general, the treatment is made much clearer and somewhat more elementary; the parts of the book given up to teaching a student what a certain procedure really measures are much amplified; more care is taken to make sure that the student understands each statistical problem itself, as well as the method to be used in solving it; the order of presentation is changed to one which experience has shown to be more convenient and illuminating to the student. I hope that it will lead whoever reads it to study modern statistical theory in far more refined and elegant presentations. To compete with any of these is the exact opposite of my intention.

Teachers College,
Columbia University,
March, 1912.

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CHAPTER I

INTRODUCTION

Mathematics and Measurements.—The power to follow abstract mathematical arguments is rare, and its development in the course of school education is rarer still. For example, few of us are able to understand the symbols or processes used in the following quotation:[1]

> The chance of r successes is greatest when r is the greatest integer in pn; this is found by the ordinary method of determining the maximum term in a binomial expansion.
>
> Let P be this maximum value $= {}^nC_{pn}\cdot p^{pn}q^{qn}$, taking the supposition for brevity that pn is integral, which will not affect the proof.
>
> $$= \frac{\underline{|n}}{\underline{|pn}\,\underline{|qn}}\,p^{pn}q^{qn}, \text{ for } pn + qn = n.$$
>
> Let P_x be chance of $pn + x$ white balls. Then
>
> $$P_x = P \times \left(\frac{p}{q}\right)^x \times \frac{qn\cdot(qn-1)\cdots(qn-x+1)}{(pn+1)(pn+2)\cdots(pn+x)}$$
>
> $$= P \times \frac{1\cdot\left(1-\frac{1}{qn}\right)\left(1-\frac{2}{qn}\right)\cdots\left(1-\frac{x-1}{qn}\right)}{\left(1+\frac{1}{pn}\right)\cdot\left(1+\frac{2}{pn}\right)\cdots\left(1+\frac{x}{pn}\right)}.$$
>
> Taking logarithms of both sides
>
> $$\log P_x = \log P + \log\left(1-\frac{1}{qn}\right) + \log\left(1-\frac{2}{qn}\right) + \cdots\cdots\cdots\cdots$$
> $$+ \log\left(1-\frac{x-1}{qn}\right) - \log\left(1+\frac{1}{pn}\right) - \log\left(1+\frac{2}{pn}\right)$$
> $$- \cdots - \log\left(1+\frac{x-1}{pn}\right) - \log\left(1+\frac{x}{pn}\right)$$

Yet this is a rather easy sample of the discussions from which the student has hitherto been expected to gain insight into the theory of measurement appropriate to the variable phenomena with which the mental sciences have to deal.

It would be unfortunate if the ability to understand and use the newer methods of measurement were dependent upon the mathematical capacity and training which were required to derive and formulate them. The great majority of thinkers would then be deprived of the most efficient weapon in investigations of mental and social facts, and adequate statistical studies could be made only by the few students of psychology, sociology, economics and education who happened to be also proficient mathematicians.

---

[1] A. L. Bowley, "Elements of Statistics," p. 275.

There is, happily, nothing in the general principles of modern statistical theory but refined common sense, and little in the technique resulting from them that general intelligence can not readily master. A new method devised by a mathematician is likely to be expressed by him in terms intelligible only to those with mathematical training, and to be explained by him through an abstract derivation which only those with mathematical training and capacity can understand. It may, neverthless, be possible to explain its meaning and use in common language to a common-sense thinker. With time what were the mysteries of the specialist become the property of all. To aid this process in the case of certain recent contributions to statistical theory is one of the leading aims of this book. Knowledge will be presupposed of only the elements of arithmetic and algebra. Artificial symbols will be used only when they are really convenient. Concrete illustrations will always accompany and often replace abstract laws.

Let no one suppose that the foregoing statements imply that mathematical gifts and training are useless possessions for a student of quantitative mental science. On the contrary, the assumption of their absence in "the reader" will necessitate long descriptions, round-about arguments and awkward formulæ. If this book were written by a mathematician for the mathematically minded it would not need to be one fifth as long. If it is read by such a one, it may well seem intolerably clumsy and inelegant.

General Information about Measurements.—There are, in addition to the recent studies of the general theory of mental measurements, a number of matters concerning the quantitative treatment of human nature which sufficient experience teaches thoughtful workers everywhere, but which have not been stated simply and conveniently in available form for study and reference. At present one must learn these gradually and with difficulty by himself, or acquire them from the oral traditions of the laboratory or classroom. They are, for the most part, extremely simple. But that one sees them at the first glance when they are presented does not imply that he would not in nine cases out of ten fail to discover them if they were not presented. To put these at the service of all who need to know about them is the second aim of this book.

The Technique of Measurements.—Although the formulæ used in expressing and computing mental measurements are in most cases straightforward and simple, they are often so foreign to the habits acquired in connection with the arithmetic and algebra of one's school days that ready and sure use of them can be acquired only by practise. Convenient and accurate manipulation of figures is one of the many things which one learns to do by doing. A mere statement of a rule leaves one uncertain. Only after applying it a number of times does he really possess it. For example, I doubt if any one of my readers is sure that from a hasty reading he understands the following, which is an accepted short method of determining the arithmetical average of a series of numbers: "Arrange the numbers in the order of their magnitude; choose any number likely to be nearest the average; add together, regarding signs, the deviations from it of all the numbers; divide this result by the number of the measures the average of which you are obtaining: add the quotient to the chosen number." To secure full mastery of every procedure taught, many model examples and sets of problems to be worked are presented.

The Application of the Theory of Measurements.—A sense of when and how to use statistical methods is even more important than knowledge of the methods themselves. The greatest benefit, therefore, will come to those who, in connection with every principle established in the text, call to mind some concrete case to which the principle should be applied. The insight into the actual use of the theory of measurement thus obtained may be increased by a critical examination of the quantitative studies referred to in Appendix I.

This book, as the title announces, deals primarily with the theory of mental and social measurements. But with a few exceptions the principles and technique which it presents are applicable to all the sciences which study variable phenomena. Physical anthropology was the first science to take advantage of them, and in medicine they will perhaps find their greatest usefulness. If one alters the language and replaces the illustrations from the realms of psychology and education by similar ones from economics, vital statistics, medicine, physiology, anthropometry or biology, as the case may be, he will find the principles to hold, with an occasional obvious modification to fit the special data. The descriptions of technical procedure similarly may, after a few obvious alterations, be applied to variable measurements in general.

The author may be permitted to express his hope that those who use the book will regard its subject matter as something more than a means to the end, convenient handling of measurements. One can use ingenuity in manipulating measurements as well as in devising experiments; can use logic in working with measures as well as in working with evidence of a more impressive and dramatic sort. Skill in expression is nowhere more required than in the task of making quantitative arguments brief, clear and emphatic. Statistics are, or at least may be, something beyond tabulation and book-keeping.

The Special Difficulties of Mental Measurements.—In the mental sciences, as in the physical, we have to measure things, differences, changes and relations. The psychologist thus measures the acuity of vision, the changes in it due to age, and the relation between acuity of vision and ability to learn to spell. The economist thus measures the wealth of a community, the changes due to certain inventions and perhaps the dependence of the wealth of communities upon their tariff laws or labor laws or poor laws. Such measurements, which involve human capacities and acts, are subject to certain special difficulties, due chiefly to (1) the *absence or imperfection of units* in which to measure, (2) the *lack of constancy in the facts* measured, and (3) the *extreme complexity* of the measurements to be made.

If, for instance, one attempts to measure even so simple a fact as the spelling ability of ten-year-old boys, one is hampered at the start by the fact that there exist no units in which to measure. One may, of course, arbitrarily make up a list of ten or fifty or a hundred words, and measure ability by the number spelled correctly.

But if one examines such a list, for instance the one used by Dr. J. M. Rice in his measurements of the spelling ability of some eighteen thousand children, one is, or should be, at once struck by the inequality of the units. Is "to spell *certainly* correctly" equal to "to spell *because* correctly"? In point of fact, I find that of a group of about one hundred and twenty children, thirty missed the former and only one the latter. All of Dr. Rice's results which are based on the equality of any one of his fifty words with any other of the fifty are necessarily inaccurate, as is abundantly shown by Table 1 (page 8).

Economists have not yet agreed upon a system of units of measurement of consuming power. Is an adult man to be scored as twice or two and a half or three times as great a consumer as a ten-year-old boy? If an adult man's consuming power equals 1.00, what is the value of that of an adult woman?

If we measure a school boy's memory or a school system's daily attendance or a working man's daily productiveness or a family's daily expenditures, we find in any case, not a single result, but a set of varying results. The force of gravity, the ratio of the weight of oxygen to the weight of hydrogen in water, the mass of the H atom, the length of a given wire—these are, we say, constants; and though in a series of measures we get varying results, the variations are very slight and can be attributed to the process of measuring. But with human affairs, not only do our measurements give varying results; the thing itself is not the same from time to time, and the individual things of a common group are not identical with each other. If we say that the mass of the O atom is sixteen times the mass of the H atom, we mean that it always is that or very, very near it. But if we say that the size of the American sibling-group is two children, we do not mean that it is that alone; we mean that it is sometimes zero, sometimes one, etc.

Even a very elaborate chemical analysis would need only a score or so of different substances in terms of which to describe and measure its object, but even a very simple mental trait—say, arithmetical ability or superstition or respect for law—is, compared with physical things, exceedingly complex. The attraction of children toward certain studies can be measured, but not with the ease with which we can measure the attraction of iron to the magnet.

The rise and fall of stocks is due to law, but not to any so simple a law as explains the rise and fall of mercury in a thermometer.

The problem for a quantitative study of the mental sciences is thus to devise means of measuring things, differences, changes and relationships for which standard units of amount are often not at hand; which are variable, and so unexpressible in any case by a single figure; and which are so complex that, to represent any one of them, a long statement in terms of different sorts of quantities is commonly needed. This last difficulty of mental measurements is not, however, one which demands any form of statistical procedure essentially different from that used in science in general.

CHAPTER II

UNITS AND SCALES

§ 1. *Common Defects in Scales for Measuring Mental and Social Facts*

Subjectivity.—When any scale of amount is used, one's natural tendency is to interpret it as we have learned to interpret the common scales for number, time, length, weight, area and the like —to regard every unit in the scale as equal to any other unit and to regard the zero of the scale as measuring *just barely not any* of the quality in question. But in the case of many of the scales used in the mental and social sciences we cannot thus take it for granted that $8 - 7 = 7 - 6 = 6 - 5 = 72 - 71 = 40 - 39$; or that *8* means twice as far from just not any of the quality in question as *4*, and *40* four times as far therefrom as *10* and half as far therefrom as *80*. The first necessity in the scientific treatment of any measure is to be aware of the meanings of the units which it represents and of the zero point from which it is reckoned.

Let us therefore examine some samples of the units and scales which have been used in the mental and social sciences. It is the custom to measure intellectual ability and achievement, as manifested in school studies, by marks on an arbitrary scale; for instance, from 0 to 100 or from 0 to 10. Suppose now that one boy in Latin is scored 60 and another 90. Does this mean, as it would in ordinary arithmetic, that the second boy has one and one half times as much ability or has done one and one half times as well? It may by chance in some cases, but the fact that the best one and the worst one of thirty boys may be so marked by one teacher, and during the next half year in the same study be marked 70 and 90 by the next teacher, proves that it need not. The same difference in ability may, in fact, be denoted by the step from 60 to 90 by one teacher, by the step from 40 to 95 by another, by the step from 75 to 92 by another, and even, by still another, by the step from 90 to 96. Obviously school marks are quite arbitrary and their use at their face value as measures is entirely unjustifiable. A 'ninety' boy may be four times or three times or six fifths as able as an 'eighty' boy.

It is the custom to measure the value of commodities and labor by their money price, but since a dollar in one year is evidently not necessarily equal to a dollar twenty years before, systems of index values have been established to give a better unit. Even these index values, as arranged by different economists, differ somewhat.

TABLE 1

THE RELATIVE FREQUENCY OF MISTAKES WITHIN THE SAME GROUP OF CHILDREN FOR EACH OF 49 WORDS TAKEN BY DR. RICE TO BE OF EQUAL AMOUNT AS MEASURES OF SPELLING ABILITY

	By 5a Grade Girls	By 5a Grade Boys		By 5a Grade Girls	By 5a Grade Boys
Disappoint	24	13	Frightened	3	6
Necessary	23	19	Baking	3	6
Changeable	20	22	Peace[1]	3	6
Almanac	19	14	Laughter	3	6
Certainly	15	15	Waiting	2	8
Lose	15	12	Chain	2	7
Slipped	13	9	Thought	2	6
Deceive	13	7	Weather	2	4
Whistling	11	11	Light	2	4
Purpose	9	10	Surface	2	4
Speech	8	15	Strange	2	4
Receive	7	12	Enough	2	2
Loose	7	7	Running	2	2
Listened	6	9	Distance	1	6
Choose	6	6	Getting	1	3
Queer	6	5	Better	1	2
Hopping	6	5	Feather	1	0
Believe	5	8	Rough	0	5
Writing	5	7	Covered	0	5
Smooth	5	5	Always	0	4
Language	5	3	Mixture	0	4
Neighbor	4	7	Driving	0	3
Learn	4	2	Because	0	1
Changing	3	11	Picture	0	0
Careful	3	8			

[1] Piece was scored correct.

For a unit of power of consumption Engel takes a child during its first year. He then calls a year-old's power of consumption 1.1; a two-year-old's, 1.2; and so on up to 3.0 for a woman 20 years or over and 3.5 for a man 25 years or over. In the United States investigation of 1890–91 the unit was taken as 100 for an adult man, 90 for an adult woman, 75 for a child 7 to 10 years old, 40 for a child 3 to 6, and 15 for a child 1 to 3. The arbitrary nature of the scale of measurement is apparent.

The inequalities of the spelling words, treated by Dr. Rice as of equal difficulty, are shown in Table 1.

The risk of accepting subjective opinion, even in the cases where it is least liable to error, may be illustrated further by the variation in judgment, even among competent authorities, as to the relative difficulty of different parts of the following simple tests:

A. How much is $\frac{\overset{12}{\cancel{144}}}{\cancel{9}} \times \frac{\overset{3}{\cancel{27}}}{\cancel{12}} \times \frac{\cancel{2}}{\cancel{9}} \times \frac{\overset{3}{\cancel{27}}}{\cancel{12}}$? - $\frac{36}{2}$ = 18

B. How much is $5\frac{3}{8} + 1\frac{1}{4} - 7\frac{1}{8} + 6\frac{1}{2}$?

C. If a girl had two dollars, three five-cent pieces, two dimes and three quarter-dollars, how much money would she have in all?

D. How much is $37\frac{1}{2} + 87\frac{1}{2} + \frac{250}{4} + 6 + \frac{1}{2} + 6$?

Twelve individuals assigned to examples *B*, *C* and *D* the amount of credit due for the successful solution of each, on the basis that the successful solution of example *A* received a credit of 10. They estimated, that is, the achievement involved in solving *B*, *C* and *D* in terms of the achievement involved in solving *A*. Their estimates varied from 8 to 20 for *B*, from 5 to 20 for *C*, and from 14 to 25 for *D*. Their ratings in detail were (Table 2):

TABLE 2

Example *B*		Example *C*		Example *D*	
Rating	Number Giving It	Rating	Number Giving It	Rating	Number Giving It
8	1	5	5	13	1
10	1	6	1	14	1
12	1	8	1	15	4
15	6	10	1	18	2
18	1	12	1	20	3
20	2	15	2	25	1
		20	1		

These variations are due to two factors; first, the variations in the opinions of the difficulty of the standard (example *A*) and, second, the variations in the opinions of the difficulty of *B, C* and *D*. We may, in part, eliminate the first factor and measure the variation which would appear if the different individuals compared their opinions of *B, C* and *D* with some objective standard, by dividing their ratings for each single example by the average of their ratings for all three. When this is done their estimates still range from 6.7 to 13.7 for *B*, from 3.0 to 10.9 for *C*, and from 10.0 to 15.5 for *D*. So, also, if we take four individuals whose ratings were such as to show that they were practically identical in their estimates of the difficulty of *A*, we find that even among just these four the ranges are 10 to 20, 5 to 15 and 15 to 25 for *B, C* and *D* respectively.

Carelessness.—In college registration statistics the unit taken is commonly one student. The college with a score of 400 is supposed to be twice as large as the college with 200. But some students do four years' work in three, while some are present only a part of the year or take only a fraction of the full course during their time of enrollment. A university with 1,000 units, made up in part of teachers taking a course or two a year, of casual students that drop out to take positions and of other irregulars, might really have a smaller attendance in the true sense, a smaller influence on students, than one with only 800 units. One person equals one person as a name or physical unit, but one person studying all his time with regular and continued attendance does not equal one person taking university work as a secondary pursuit.

In measuring the fertility, or rather the reproductivity, of human beings, it seems at first thought to be justifiable to use the number of children in the family as a measure. But is not the number of children who live a better measure? And may not the number of children who live through the reproductive period (say, fifty years) be a still better measure? And is not, perhaps, the number of children, each weighted in some way by the length of his life, another measure to be considered? Surely a child who dies in five minutes is not equal as a measure of reproductivity to a child who lives sixty years. Is a child who lives only thirty years?

In the case of the "college student" and the "child born" we are misled by what Professor Aikins has called the "jingle" fallacy. The *words* are identical and we tend to accept all the different things to which they may refer as of identical amount. A similar unthinking acceptance of verbal equality as a proof of real equality makes one measure labor on the hypothesis that any one hour is equal to any other hour of it, forgetting that the step from 7 to 8 hours *per diem* may be different from the step from 8 to 9 and is obviously different from the step from 20 to 21 hours. The fallacy may be emphasized by one final illustration. Dr. Swift, in studying the effect of practise, measured motor skill by the number of time two balls could be kept tossed in the air with one hand. He took as a unit of measurement one successful pair of tosses and regarded any one such pair as equal to any other. For him, that is, the step from 0, or inability to catch and toss again at all, to 5, or the ability to catch and toss 5 times with each ball, is equal to the step from 200, or ability to keep the balls in the air 200 times without failure, to 205, or the ability to do so 205 times. But, of course, if one can toss the balls 200 times, he can, so far as motor skill goes, toss them 205 times almost as easily, the step being nearly zero. On the other hand, the step from 0 to 5 is a very considerable gap, one which some individuals can never pass. The result of Dr. Swift's system of units is that he gets the appearance of very slow improvement in early hours of practise and very rapid improvement in late hours, a state of affairs which contradicts what is found by other investigators. Of course, "tossing two balls once" sounds identical with "tossing two balls once," but it is not.

§ 2. *The Essentials of a Valid Scale*

Objectivity.—What science means by a perfectly "objective" scale is a scale in respect to whose meaning all competent thinkers agree. A perfectly "subjective" scale is one in respect to whose meaning all competent thinkers disagree (save by chance). These are limits between which the actual scales known to science lie. Near the former extreme is the scale of length,—*one, two, three, . . . , n millimeters* being understood by competent thinkers to be certain multiples of a certain rod kept in Paris at a certain temperature, or certain multiples of the wave-length of cadmium light. Near the latter extreme is the following scale: *possessing zero, or just not any, beauty, very beautiful, extremely beautiful.* If a thousand competent students of esthetics should state just what each understood "*extremely beautiful*" to mean in terms of a fact which they could all observe and identify, they would disagree widely. One might say, for example "as beautiful as Milton's sonnet on his blindness;" another, "as beautiful as Rembrandt's Mill;"another, "as beautiful as the Parthenon;" and so on. Only by chance would any two hit upon the same observable fact to represent their meaning.

In between these extremes are all degrees of objectivity—that is, expert agreement as to the meaning of the scale. Thus, the scale of cloudiness ranging from *0, or a perfectly clear sky*, to *10, or as cloudy a day as is experienced,* would be more objective than the scale for beauty given above, since expert meteorologists would agree better in the observable facts which they would choose to express their interpretations of *0, 1, 2,* and the rest, than artists would in their choices for zero beauty and the rest. Thus the scale of illuminating power is somewhat less objective than the scale of length, but much more objective than the scale of cloudiness. A thousand expert illuminating engineers thinking of "eight candle power" would agree less than the physicists would about eight millimeters, but more than the meteorologists would about eight degrees of cloudiness.

Scales in respect to whose meaning competent thinkers could agree rather closely were devised early in the case of number, time, length and weight. Similar scales for temperature, heat, force and "value in exchange" came later. Similar scales for measuring intellectual maturity, the standard of living, ability in prose composition and other facts of intellect, character and social condition are being devised now.

The gain for thought and practise that comes from the mere definition of words that have been used vaguely and loosely as a crude scale is extraordinary. Suppose, for example, that students of esthetics made plates of twenty drawings ranging from very inferior up to excellent ones, from which identical series of plates and prints could be reproduced as we reproduce our millimeter, centimeter and the like from the rods in Paris. Suppose that the terms *a, b, c, d, e, f, g, h, i, . . . , t,* were used universally to refer to the amounts of beauty possessed respectively by these twenty prints. Artists, teachers of drawing, critics and dealers could, by this very easy device, each know what any other meant when he gave an estimate of the general beauty of a drawing. Instead of unintelligible rhetoric or elaborate searching for some comparison to make his meaning clear, the speaker or writer could define what degree of beauty he meant as d, e or s, as he now defines the size of a drawing, the age of the man who made it, or the price at which it last sold. To replace the crude and vague comparatives and superlatives and other words descriptive of different amounts of various mental and social facts by references to scales of accepted meaning in terms of observable facts is indeed one of the first and greatest duties of the mental sciences.

Consistency.—The series of facts used as a scale must be varying amounts of the same sort of thing or quality. This requirement needs no comment.

Definiteness of the Facts and Their Differences, One from Another.—An ideal scale, such as that for weight, is a series of perfectly defined amounts, the differences between any two of them being also perfectly defined, so that a series varying by steps of equal difference can readily be selected.

It is not necessary, however, to have a scale arranged in equal steps, though it is very desirable. It is not even necessary to know whether or not the steps are equal, though it is very desirable to know it, and, if the steps are unequal, to know the exact degrees of the inequalities.

A scale in the sense of a series of defined and accepted facts with which other facts may be compared is useful. If the mere order of magnitude of these facts is known the scale is still more useful. If the steps of difference are known to be equal or to be unequal, new utilities are created. If the amount of inequality in each case is roughly known, so much the better; if precisely, so much the better. If the steps are approximately equal, so much the better; for them to be exactly equal is best of all.

These facts may be thought of conveniently in algebraic form. Suppose, in the case of drawings, that we have merely a series of defined facts, so that say drawings to be measured in beauty may be called equal to a or equal to b. We can define the beauty x of a drawing as, $x = a$. Similarly we can know that $v = d$, $w = m$, $z = c$, and the like. If $y = a$, we can infer that $x = y$.

Suppose now that the mere *order* of magnitude of the facts is known so that the order of the alphabet is the order from least to most beauty. We have then $a < b < c < d < e \dots < s < t$. If $x = a$, $v = d$, $w = m$, and $z = c$, we can infer that $x < z$ or v or w, that $z < v$ or w, that $v < w$, and the like; that $w - z > w - v$, that $w - x > z - x$, and the like. Suppose that one knows further that the steps are *not* equal. The possibility of wrongly assuming equality by analogy with other scales is thereby prevented. We know, for instance, that $v - x$ need not be three times $v - z$, that $w - x$ need not be twelve times $v - z$ or six times $z - x$, and the like.

Suppose that the amount of inequality in each case is known. We then have the values of a, b, c, etc., all placed correctly on a scale so far as concerns all the relations of the distances between any two to the distance between any other two. That is, we have, letting K stand for the difference $b - a$ and letting the letters α, β, γ, ∂, etc., stand for known fractions or multiples of unity:

$$\begin{aligned} b - a &= K, \\ c - b &= \alpha K, \\ d - c &= \beta K, \\ e - d &= \gamma K, \\ f - e &= \partial K, \text{ etc.} \end{aligned}$$

We may then know that, if $x = a$, $v = d$, $z = c$, as before,

$$\begin{aligned} z - x &= \alpha K + K, \\ v - x &= \beta K + \alpha K + K, \\ v - z &= \beta K, \\ v - x &= \text{a known multiple of } v - z, \text{ or of } z - x, \text{ etc., etc.} \end{aligned}$$

Suppose finally that the differences are all equal. The relations just mentioned then all become convenient integral multiples of the difference between any one drawing and the drawing next to it in the scale series.

$$\begin{array}{llll} b - a = K, & c - b = K, & d - c = K, & \text{etc.,} \\ b - a = K, & c - a = 2K, & d - a = 3K, & \text{etc.,} \end{array}$$

so that

$$\begin{aligned} z - x &= 2K, \\ v - x &= 3K, \\ v - z &= K, \\ v - x &= 3(v - z) \text{ or } 1\tfrac{1}{2}(z - x), \text{ etc., etc.} \end{aligned}$$

Comparability with the Facts to be Measured.—The time and skill required for comparing or matching a fact with the scale by which it is to be measured varies greatly, according to the scale. Other things being equal, it is much harder to measure the length of a man's head with an ordinary foot-rule alone than with calipers and a foot-rule. To measure the beauty of a drawing of an eagle by comparing it with the series *a*, *b*, *c*, etc., just described and observing to which point of the series it was nearest in respect to beauty, would be much easier if the series consisted of drawings of the same eagle, than if the series consisted of drawings of ships.[2]

The accuracy with which a fact can be matched with its proper point on a scale also varies greatly with the scale. If the observer gave equal time and effort to the task in both cases he would make a larger error in measuring head-length with the ordinary foot rule than if calipers also were used; and a larger error in defining the beauty of the drawing of the eagle by the series of prints of ships than by the series of drawings of eagles.

This fact of the varying difficulty, as to time, skill and precision, in the use of different scales is often distorted into the false notion that scales are of two sharply separated sorts—scales whose use does not depend at all, and scales whose use does depend greatly, on the observer using them. Really the differences are continuous gradations. Just as there is a continuous range from little to much agreement in respect to the meaning of terms or points on a scale, so there is in respect to the time or skill required and the precision obtained. In particular, to distinguish scales that can be used for comparison with other facts "objectively" in the sense of "with perfect agreement amongst competent observers" from scales that can be used only "subjectively" in the sense of with a large disagreement or set of "personal equations" is very misleading. No comparison of anything in nature with anything else is errorless.[3] And every comparison of anything in nature with anything else is subject to an error if the facts are specimens of continuous quantities like length, time, permanence in memory, beauty, and the like. With the best instruments to measure the weight of a given cannon-ball the best hundred experts would not agree within one hundred-thousandth of a milligram. With the best thermometers they could not do so well for temperature. Personal equations always enter if the distinctions required are made fine enough. The process of matching sticks with a scale for length is logically and statistically the same as that of matching drawings with a scale for beauty. The disagreements, the lack of precision, would merely be a few thousand or million times as great in the latter case.

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[2] Other things being equal.

[3] Simple counting or comparing the number of objects in a given collection with a series of integral numbers to locate the number that fits the number of objects in the collection may seem to be an exception, but it is not. The hundred most competent observers living would not always agree in their counts of a thousand barrels of pennies.

Minima for the time, effort and skill required and the errors made in matching a fact with a scale are then, though very desirable features of a scale, in no sense necessary. Scales for mental and social measurements can be of great service in spite of gross inferiority, in these respects, to the common scales for physical facts.

Reference to a Defined Zero Point.—Finally one must know what fact would, by the scale as used, be measured as zero or just barely above zero. The zero-point may be *absolute*, meaning "just not any of" the thing, or *arbitrary*, meaning a point *called* zero though actually designating some amount of the thing. Thus the thing being temperature, 20° C. is 20 degrees above the arbitrary zero—the melting point of ice—and 293 degrees above the supposed absolute zero of just not any molecular motion in a gas.

This last requisite for a valid scale requires further comment. In the physical sciences, we can find or infer the place where a certain thing begins—the least amount of length, or mass, or velocity, or resistance and the like. Such absolute zero-points are indeed often obvious to any one. But absolute zeros for goodness, intellect, delicacy of discrimination, memory, quickness, courage, inventiveness, and the like are never obvious and, for the most part, are undiscovered. When one says that four pounds is "two times as heavy as," "or two times as much mass as," two pounds, he and his hearers know that he means that the former is represented by a point on the scale for weight twice as far from "just not any mass" as is the latter. But a similar proof that *A*'s delicacy of discrimination of length is twice *B*'s, or that *A* has three times as much courage as *B*, is at present impossible. What "just not any delicacy of discrimination" or "just not any courage" is, must first be discovered.

When absolute zero points are not available, it is imperative to consider what the arbitrary point is from which the scale in use starts. Thus, in the case of delicacy of discrimination of length, what is actually done is to measure on a scale of *amount of error made*, zero meaning the limit of perfect discrimination, or on a scale of difference required for discrimination, zero meaning, as before, the limit of *perfect* discrimination. In the case of courage, what we in fact do is to calculate from a vague notion of zero, either as very little courage or as the courage of the average man.

When the zero point has to be chosen arbitrarily it is well worth while to consider the meaning and utility of each of the different possible ones. Other things being equal, a point somewhere near "just not any of the trait in question" has great advantages over a point well up on the scale, such as the condition of the average man in the trait.

The influence of the zero point of a scale upon measurements made by that scale will alter the interpretation of, but not the method of making, measurements of things and conditions; but when things or conditions are *compared*, that is, when measurements are made of differences, changes and relations, it becomes of the utmost importance. For, in the case of differences, changes and relations, it is usually desirable to be able to use the 'times as ——' comparison. But such comparisons are subject to momentous misunderstandings unless the zero points are absolute. One of the common fallacies in the mental sciences is to compare directly the amounts of measurements made from different zero points. Another is to use arbitrarily some point along the scale as if it were an absolute zero point. Silly as it may appear, we often with mental measurements do such arithmetic as the following:

"John, who weighed 4 lbs. more than 100 lbs., has added 2 lbs. to his weight; James, who weighed 100 lbs. more than 10 lbs., has added to his weight 50 lbs. Both gained 50 per cent. and so their relative gains were equal."

"John weighs 10 lbs. more than 60 lbs. James weighs 2 lbs. more than 60 lbs. John is five times as heavy as James."

It should be obvious that the discovery of even a rough approximation to an absolute zero point for any scale is of great advantage to thought and practise. For example, in the supposed case of a scale for beauty of drawings, as soon as we have found such a point—that is, a drawing of approximately just not any beauty—and have stated the difference between it and any one of the drawings of the scale in terms of any unit of the scale, all the facts of the scale become amenable to ordinary arithmetical procedure, including the "times" judgment. Thus, suppose a drawing, *u*, to be found of approximately zero beauty and six times as far below *a* as *a* is below *b*. Then the drawings *a, b, c, d*, etc., of the previous illustration can be renamed 6, 7, 8, 9, etc., and, with only a very slight error, treated as we treat inches, dollars or pounds.

The scales in actual use in psychology, education, sociology, history and the like are often inadequate in respect to one or more of the essentials of a scale. The work of the student of mental and social measurements is then, to replace them by better ones so far as he can, to devise methods to make the most out of those which he does not replace, and to avoid attributing to a measurement properties which the scale by which it was obtained does not justify. The last two tasks need no further mention at this point. Concerning the first, it has already been suggested that in cases where quantitative study of human nature and achievement is balked at the very beginning by the lack of series of defined amounts, whose differences from each other and from defined zero points are known, this lack is due rather to lack of study than to any essential insusceptibility of human behavior to rating in units of amount on intelligible scales. The following foot-rule for merit in English composition may serve as an illustration of the principle that any varying facts which can be estimated at all in terms of the amount of some one thing, can be measured in terms of defined units whose distances from a defined zero are known. I quote this scale without any justification of its choice of a zero point, or of the facts taken to represent differences of 18, 26, 37, 47 and so on from that zero point. Such justification will be found in a full account of the scale by its author, Professor M. B. Hillegas, soon to be published.

§ 3. *A Sample Scale*

A Scale for Merit in English Composition by Young People

0. Dear Sir: I write to say that it aint a square deal Schools is I say they is I went to a school. red and gree green and brown aint it hito bit I say he don't know his business not today nor yeaterday and you know it and I want Jennie to get me out.

18. the book I refer to read is Ichabod Crane, it is an grate book and I like to rede it. Ichabod Crame was a man and a man wrote a book and it is called Ichabod Crane i like it because the man called it ichabod crane when I read it for it is such a great book.

26. Advantage evils are things of tyranny and there are many advantage evils. One thing is that when they opress the people they suffer awful I think it is a terrible thing when they say that you can be hanged down or trodden down without mercy and the tyranny does what they want there was tyrans in the revolutionary war and so they throwed off the yok.

Sulla as a Tyrant

37. When Sulla came back from his conquest Marius had put himself consul so sulla with the army he had with him in his conquest seized the government from Marius and put himself in consul and had a list of his enemys printy and the men whoes names were on this list we beheaded.

De Quincy

47. First: De Quincys mother was a beautiful women and through her De Quincy inhereted much of his genius.

His running away from school enfluenced him much as he roamed through the woods, valleys and his mind became very meditative.

The greatest enfluence of De Quincy's life was the opium habit. If it was not for this habit it is doubtful whether we would now be reading his writings.

His companions during his college course and even before that time were great enfluences. The surroundings of De Quincy were enfluences. Not only De Quincy's habit of opium but other habits which were peculiar to his life.

His marriage to the woman which he did not especially care for.

The many well educated and noteworthy friends of De Quincy.

Fluellen

58. The passages given show the following characteristic of Fluellen: his inclination to brag, his professed knowledge of History, his complaining character, his great patriotism, pride of his leader, admired honesty, revengeful, love of fun and punishment of those who deserve it.

Ichabod Crane

67. Ichabod Crane was a schoolmaster in a place called Sleepy Hollow. He was tall and slim with broad shoulders, long arms that dangled far below his coat sleeves. His feet looked as if they might easily have been used for shovels. His nose was long and his entire frame was most loosely hung to-gether.

Going Down with Victory

77. As we road down Lombard Street, we saw flags waving from nearly every window. I surely felt proud that day to be the driver of the gaily decorated coach. Again and again we were cheered as we drove slowly to the postmasters, to await the coming of his majestie's mail. There wasn't one of the gaily bedecked coaches that could have compared with ours, in my estimation. So with waving flags and fluttering hearts we waited for the coming of the mail and the expected tidings of victory.

When at last it did arrive the postmaster began to quickly sort the bundles, we waited anxiously. Immediately upon receiving our bundles, I lashed the horses and they responded with a jump. Out into the country we drove at reckless speed—everywhere spreading like wildfire the news, "Victory!" The exileration that we all felt was shared with the horses. Up and down grade and over bridges, we drove at breakneck speed and spreading the news at every hamlet with that one cry "Victory!" When at last we were back home again, it was with the hope that we should have another ride some day with "Victory."

Venus of Melos

83. In looking at this statue we think, not of wisdom, or power, or force, but just of beauty. She stands resting the weight of her body on one foot, and advancing the other (left) with knee bent. The posture causes the figure to sway slightly to one side, describing a fine curved line. The lower limbs are draped but the upper part of the body is uncovered. (The unfortunate loss of the statute's arms prevents a positive knowledge of its original attitude.) The eyes are partly closed, having something of a dreamy langour. The nose is perfectly cut, the mouth and chin are moulded in adorable curves. Yet to say that every feature is of faultless perfection is but cold praise. No analysis can convey the sense of her peerless beauty.

A Foreigner's Tribute to Joan of Arc

93. Joan of Arc, worn out by the suffering that was thrust upon her, nevertheless appeared with a brave mien before the Bishop of Beauvais. She knew, had always known that she must die when her mission was fulfilled and death held no terrors for her. To all the bishop's questions she answered firmly and without hesitation.

The bishop failed to confuse her and at last condemned her to death for heresy, bidding her recant if she would live. She refused and was led to prison, from there to death.

While the flames were writhing around her she bade the old bishop who stood by her to move away or he would be injured. Her last thought was of others and De Quincy says, that recant was no more in her mind than on her lips. She died as she lived, with a prayer on her lips and listening to the voices that had whispered to her so often.

The heroism of Joan of Arc was wonderful. We do not know what form her patriotism took or how far it really led her. She spoke of hearing voices and of seeing visions. We only know that she resolved to save her country, knowing though she did so, it would cost her her life. Yet she never hesitated. She was uneducated save for the lessons taught her by nature. Yet she led armies and crowned the dauphin, king of France. She was only a girl, yet she could silence a great bishop by words that came from her heart and from her faith. She was only a woman, yet she could die as bravely as any martyr who had gone before.

§ 4. *Technical Details Concerning Scales*

Discrete and Continuous Series.—Quantities to be measured may be in a *discrete* or in a *continuous* series. A *discrete* series is one with gaps. Thus if we measure the number of children in a class we can get only integral numbers. Sixth tenths of a man, ninety-two hundredths of a man, do not exist. There are gaps, between one man and two, two men and three, etc. A *continuous* series, such as time or velocity or intellect or wealth, is in theory capable of any degree of subdivision. Almost all mental traits and social facts due to human action are quantities in continuous series.

Any given measure of a continuous series means not a single point on the scale of measurement, but the distance along that scale between two limits. Thus if we measure the time taken to perceive and react to a signal in thousandths of a second and get .143 sec. as the measure, the .143 means commonly that that was the nearest point, that the time was nearer to .143 than to .142 or to .144; and this means, of course, that the time was between .1425 and .1435. The truer statement would be, "*A*'s reaction time is between .1425 and .1435." If we measure a man's wealth in dollars as 73,448, we do not mean that he has exactly that, but that that is the nearest dollar mark. At times a measure does not mean that the individual to whom it is given is nearer to that measure than to any other on the scale used, but that he is *above it and not up to the next* measure. For instance, if a boy in 10 minutes gets the answers to 5 problems in arithmetic, we would commonly score him 5, but our 5 would mean, "at least 5 and not 6." The boy might, for instance, have almost completed the sixth in his mind, and really be, if we had a finer scale, 5.9. In mental measurements, any figure—say, 21—may mean between 20.5 and 21.5, or between 21 and 22. It might also mean between 20 and 21, if we measured people by the point which they just did not reach, but this is almost never a useful method. The second method of measuring by the last point on the scale passed is in many mental traits the natural one and often saves labor in all sorts of measurements.[4]

In later operations with figures denoting measurements the method of obtaining them and their consequent meaning must be kept in mind. If each one of a set of measures means "from this number to the next on the scale," then the average calculated from them will, to represent a point on the scale, need to be increased by .5 the unit of the scale. A little experimentation and thought will create the useful habits of thinking of any number for a measure on a continuous scale as representing the quantities between two limits; of realizing that, for our ordinary arithmetic, it represents the space from a point half-way between it and the number below to a point half-way between it and the number above; and of understanding that if our method of measurement makes it represent some other space, we must make proper allowance in calculation.

Undistributed Measures.—In many continuous series the measure 0 (zero), which should mean a definite distance on the scale, either from $-.5$ to $+.5$, or from 0 to .1 on the scale, means only an indefinite distance; namely, from a point above 0 to an unknown lower extreme. Thus, if, in measuring arithmetical ability by a test of 20 examples, we should find out of fifty boys a dozen who did none at all and should mark them zero, we could not assume that they were as a group the same distance below the '1 to 2' group as the '1 to 2' group were below the '2 to 3' group. All that would be known about the dozen boys would be that they belonged somewhere below 1. One of them might be really as far below a boy marked 1 as the latter was below a boy marked 20. In such cases we call the zero marks *undistributed* or indefinite. The same holds good, of course, for the upper as well as the lower extreme. If, in the illustration in question, a dozen boys had done all the examples perfectly and been marked 20, that score would mean, not that the boys were between 20 and 21, but that they were somewhere above 20. One should always guard against undistributed measures at either extreme of a scale.

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[4] It is easier to put a measure between two points on the scale than to tell to which point it is nearest. Moreover, in dropping insignificant figures it is easier to drop absolutely than to add one unit to a given 'place' when the figure dropped is over .5 the unit of the next place.

The Interpretation of Measures.—In using measures recorded by others, it is necessary to know by what method and to how fine a degree the measurements were made. Thus, suppose one worker, *A*, is using the scale for English composition to grade specimens to the nearest point on the scale, '0, 18, 26, 37, etc.'; suppose another, *B*, to grade them as nearest to the scale points, *or to imagined qualities halfway between*—that is, as 0, 9, 18, 22, 26, 31½, 37, 42, 47, etc.; suppose a third, *C*, to grade the specimens as between the limits, 0 to 18, 18 to 26, 26 to 37, 37 to 47, etc., using 0, 18, 26, 37, etc., in these meanings; suppose a fourth, *D*, to grade to a single unit, letting 0, 18, 26, 37, etc., stand for qualities indistinguishable from the qualities shown on the scale and letting 0, 1, 2, 3, 4, 5, 6, 7, 8, etc., stand for qualities between 0 and 18, etc.

The grade of 18 would then mean from 9 to 22 if given by *A*; from 13.5 to 20 if given by *B*; from 18 up to 26 if given by *C*; and presumably from 17.5 to 18.5 if given by *D*.

In the same way a measure of 50 centimeters may mean from 40 up to 60, from 49.5 up to 50.5, from 50 up to 51, or from 49.95 up to 50.05. A frequent error is to put the measures 50.5, 50.6, 50.7, 50.8, 50.9, 51.0, 51.1, 51.2, 51.3, and 51.4 (all in centimeters), measures being taken to the nearest millimeter, when grouped, as equal to the measure 51 cm., measures being taken to the nearest centimeter. They are not, of course, since they cover the space from 50.45 up to 51.45, while the latter covers that from 50.5 up to 51.5.

§ 5. *Measurement by Relative Position*

Many mental phenomena elude altogether direct measurement in terms of amount. How many thefts equal in wickedness a murder? If the piety of John Wesley is 100, how much is the piety of St. Augustine? How much more ability as a dramatist had Shakespeare than Middleton? What per cent. must be added to the political ability of the Jewish race to make it equal to that of the Irish race? In these and similar cases the quality to be measured manifests itself objectively in so complicated and subtle effects that the task of expressing it in units of amount is almost hopeless.

Nevertheless, such phenomena can be measured and subjected to exact quantitative treatment. Though we can not equate crimes, we can arrange them in a list according to their magnitude, and measure any one by its position in the list. Similarly St. Augustine, if placed in his proper rank amongst men for piety, is measured as exactly as if given a numerical score. The step from Shakespeare to Middleton in a series of dramatists ranked in order of ability is a definite measure. If a boy moves in English composition from the position of the 500th in a thousand to the position of the 74th in a thousand his gain is measured as clearly and exactly as when we measure the inches he has grown in height. Measurement by relative position in a series gives as true, and may give as exact, a means of measurement as that by units of amount. Measurement by relative position in scientific studies is of course but an outgrowth of the common practise of mankind. The man in the street measures things not only as being so many times this, but also as being "the biggest he ever saw" or "about average size."

Measures by amount of some unit have been the subject of great development in the hands of physical science, while measures by relative position have been comparatively neglected, though for the mental sciences they are of the utmost importance. The use that has been made of them already by Galton, Cattell and others gives promise that the value of a measure to which the most subtle and the most complex traits alike are amenable will in the future be more appreciated.

In measuring any product or person by position in a series, the chief desiderata are:

1. That the arrangement of the series should not be the result of

any individual's chance bias, *i. e.*, that the arrangement should represent the general tendency of a number of observers.

2. That it should not be influenced by a constant error, by bias common to all, *i. e.*, that there should be, on the whole, as much bias in any one direction as in any other.

3. That it should be on a sufficiently minute scale,—that is, include a large enough number of 'groups' or 'grades' or 'ranks.'

Suppose, for instance, that we wish to find the position of a certain drawing amongst a thousand made by first-year high-school boys. No one person can, except by accident, be a perfect rater of these, for his momentary impulse or his peculiar ideals or training will overweight certain features. The combined opinion of ten equally good judges will always be truer than the opinion of any one of them. If, however, all the ten over-emphasized color or perspective, their combined rating would be false. Such a constant error in judgment is avoided as far as possible if judges are chosen at random from amongst those esteemed competent.

The value of having the drawings arranged on a fine scale is: first, that the finer the scale the more precise the measure, and, second, that if a drawing is then misplaced by chance it will not be displaced so far. For instance, if drawings were rated simply *Good* or *Bad*, one near the dividing line, if put on the wrong side, would be put very far to the wrong side, *viz.*, one fourth of the total distance, whereas if they were rated in twenty divisions, one in the middle would, if put to the wrong side, be moved only one fortieth of the total distance. As a practical rule one should divide the series into as many groups as one can distinguish.

Amongst school abilities, achievements in handwriting, drawing, painting, writing English, translation, knowledge of history, geography, etc., are readily measured by serial rating and the agreement of competent observers is such that great reliance can be put upon the results from the ratings of, say, twenty such. In the case of more general characteristics the service of the method will be greater still, though the readiness and accuracy of the process are less.

Measures by relative position have one grave defect. Ordinary arithmetic does not apply to them. It is not possible to add '17th from top of 1,000 in wealth' to '92d from top of 1,000' as we can add 'fortune of $1,000,000' to 'fortune of $790,000.' We cannot say that the 10th ability from the top in 100 plus the 20th ability from the top in 100 is equal to the 14th plus the 16th. We can not equate different positions in the series with each other as we can different amounts of the same thing.

We can not, that is, on the basis of what has been so far said about measurement by relative position in a series. There are, however, two possibly valid ways of transmuting a measure in terms of relative position into terms of units of amount. Given a certain condition of the series as a whole, and the statements of position can be expressed in terms of amount and made amenable to ordinary arithmetic. Given the truth of certain theories of the relation of the amount of difference to the ease of observing it, and the same result will hold. These possibilities will be discussed in a special chapter on measurement by relative position.

PROBLEMS

1. Why would the number of men giving instruction in a university not be a fair measure of the amount of teaching done?

2. What are the faults of the following proposed as a measure of civilization: $\frac{\text{Birth-rate}}{\text{Death-rate}}$?

3. How could you get commensurate units of amount of ability in addition? In what sense could you, after obtaining such units, say that *A*'s ability in addition was twice or three times *B*'s?

4. In giving examination marks, the custom is to measure downward from a standard of perfection. Suggest a better starting point to take.

5. Consider each of these ten sets of measures. Describe each in respect to its (1) objectivity, (2) equality of units, and (3) zero point.

MEASUREMENTS OF THREE INDIVIDUALS, *A*, *B*, AND *C*

	A	*B*	*C*
I. Stature	160 cm.	140 cm.	130 cm.
II. Simple reaction-time to sound	.175 sec.	.125 sec.	.150 sec.
III. Average error in drawing a line to equal a 100 mm. line	3.2 mm.	2.8 mm.	2.2 mm.
IV. Number of words (of a list of 12, heard at a rate of 1 per second) remembered long enough to write them immediately after the last word was read	6 words	9 words	7 words

V.	Quality, or merit, or goodness of handwriting	illegible	legible	perfect
VI.	School marks in spelling	82	62	93
VII.	Efficiency in perception; the number of A's marked in 60 seconds on a sheet containing 100 A's mixed with 400 other capital letters	48 A's	60 A's	82 A's
VIII.	Criminality: number of times convicted of a penal offense	0	1	0
IX.	Degree of interest in music	little	moderate	a great deal
X.	Age in days	5,080 d.	6,150 d.	5,615 d.

6. Group the following measures by whole numbers, first, by using the whole numbers 14, 15, etc., to represent 13.5–14.499, 14.5–15.499, etc., and second by using 14, 15, etc., to represent 14–14.999, 15–15.999, etc.:

18.642, 17.39, 21.45, 14.81, 15.51, 17.23, 19.60, 18.42, 21.7,
15.861, 16.5, 17.92, 14.4, 19.38, 20.6, 20.5, 18.39, 17.489.

Which method would you expect to be the easier and least subject to error if one had equal amounts of practise with both? Why?

7. Name five things or qualities or traits the different amounts of which vary by discrete steps. Name five which vary continuously. Name five which are now ordinarily measured by relative position.

CHAPTER III

THE MEASUREMENT OF A VARIABLE FACT

§ 6. *Tables and Surfaces of Frequency*

ANY fact in any human being or institution is a variable quantity. If we measure it a number of times with a fine enough scale of measurement we get, not one constant result, but many differing results. The amount of addition John Smith can do in a minute, the number of cubic feet of sand Tom Jones can dig in an hour, the food consumed by Richard Brown in a day, the weekly earnings of a particular factory—these and all facts depending on human nature and behavior are variable.

The Total Distribution of a Fact.—A constant can be measured in a single number, but a variable for its complete measurement requires as many different numbers as there are varieties of the thing. Since John Smith can add now 20, now 21, now 22, now 23 digits in a minute, his ability is not any one of these nor the average of them all, but is described truly only as "20 such and such a per cent. of the times, 21 such and such a per cent. of the times," etc. Any single number would be but an extremely inadequate representation of his ability in addition or of that of any variable trait. The measure of a variable quantity implies a list of the different quantities appearing, with a statement of the number of times that each appeared. Such a list and statement together are called a *table of frequencies* or a *distribution* of a trait. The measure of a variable fact is thus its entire distribution or table of frequency. Tables 3, 4 and 5 thus measure the three facts denoted by their titles.

It is common to present a table of frequencies in a diagram in which distances along a line represent the different quantities, and the heights of columns erected along it their frequencies. Thus Figs. 1, 2 and 3 represent at once to the eye the facts given by Tables 3, 4 and 5. Such a figure is called a *surface of frequency;*

TABLE 3

MEMORY SPAN OF B. F. A.

Of a series	of 12 letters read	1 was	correctly written	and placed	2 times	or in	5%
"	"	2 were	"	"	4	"	10
"	"	3 "	"	"	3	"	7.5
"	"	4 "	"	"	7	"	17.5
"	"	5 "	"	"	6	"	15
"	"	6 "	"	"	9	"	22.5
"	"	7 "	"	"	0	"	0
"	"	8 "	"	"	6	"	15
"	"	9 "	"	"	2	"	5
"	"	10 "	"	"	1	"	2.5

There were 40 trials in all.

TABLE 4

ACCURACY OF DISCRIMINATION OF LENGTH OF E. H.

In drawing	a line to equal	a 100-mm. line	an error of	− 7	mm. occurred	2	times.
"	"	"	"	− 6	"	2	"
"	"	"	"	− 5	"	6	"
"	"	"	"	− 4	"	8	"
"	"	"	"	− 3	"	7	"
"	"	"	"	− 2	"	8	"
"	"	"	"	− 1	"	11	"
"	"	"	"	0	"	13	"
"	"	"	"	+ 1	"	11	"
"	"	"	"	+ 2	"	7	"
"	"	"	"	+ 3	"	3	"
"	"	"	"	+ 4	"	5	"
"	"	"	"	+ 5	"	8	"
"	"	"	"	+ 6	"	4	"
"	"	"	"	+ 7	"	1	"
"	"	"	"	+ 8	"	1	"
"	"	"	"	+ 9	"	1	"
"	"	"	"	+10	"	2	"

There were 100 trials in all; hence per cents. = times.

TABLE 5[1]

PER CENT. PER YEAR OF MEMBERS OF THE AMALGAMATED SOCIETY OF ENGINEERS IN WANT OF EMPLOYMENT DURING 31 YEARS

Less	than	1 %	lacked	employment in	1	out of	31 years,	3.2 %
1 %	to	2 %	"	"	8	"	"	25.8
2	"	3	"	"	4	"	"	12.9
3	"	4	"	"	4	"	"	12.9
4	"	5	"	"	4	"	"	12.9
5	"	6	"	"	2	"	"	6.5
6	"	7	"	"	5	"	"	16.1
7	"	8	"	"	2	"	"	6.5
8	"	9	"	"	1	"	"	3.2
9	"	10	"	"	0	"	"	
10	"	11	"	"	0	"	"	
11	"	12	"	"	0	"	"	
12	"	13	"	"	0	"	"	
13	"	14	"	"	1	"	"	3.2

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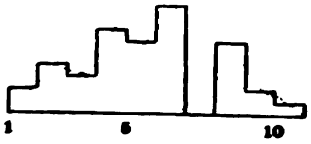
[1] Arranged from data given by George H. Wood on pages 640–642 of Vol. 62 of the *Journal of the Royal Statistical Society*.

the compound line which, with the horizontal base line, encloses it, is called a *distribution curve.* Another method of presenting graphically a table of frequencies is to draw instead of the top lines of the columns a line joining the middle points of these top lines. Figs. 1A, 2A and 3A repeat Figs. 1, 2 and 3 in this form.



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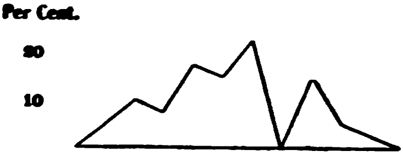
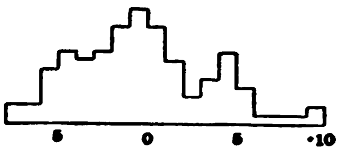


1A



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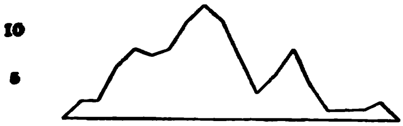
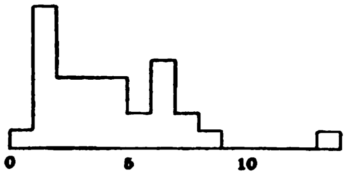


2A



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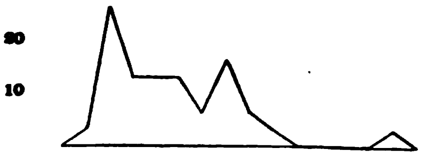


3A

FIG. 1. Surface of frequency of the ability of B. F. A. in memory span. Number of letters correctly written and correctly placed, after one hearing of a series of 12. Number of measurements = 40.

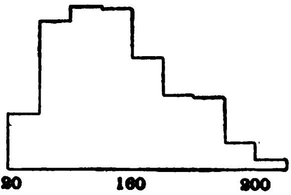
FIG. 2. Surface of frequency of the ability of E. H. in discrimination of length. Number of millimeters error made in drawing a line to equal a 100-mm. line. Number of measurements = 100.

FIG. 3. Surface of frequency of the opportunity for work in a trade. Number of members of the Amalgamated Society of Engineers lacking employment. Number of measurements = 31 (years).

FIG. 1A. Same as 1, but drawn by joining mid-points of columns.

FIG. 2A. Same as 2, but drawn by joining mid-points of columns.

FIG. 3A. Same as 3, but drawn by joining mid-points of columns.



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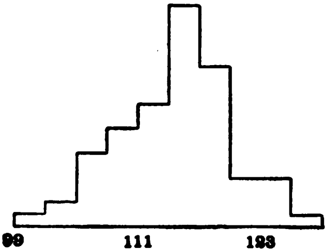

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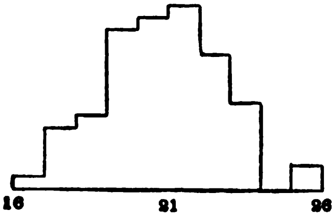

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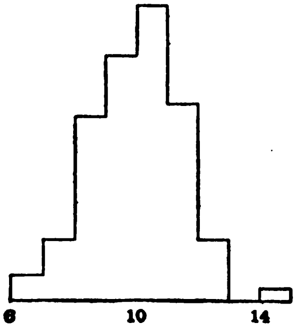

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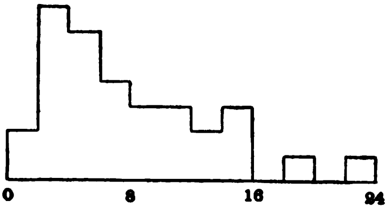

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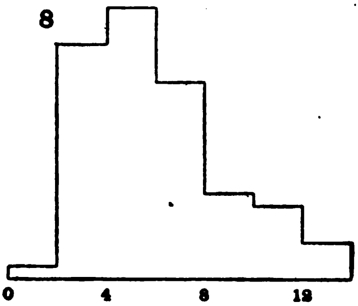

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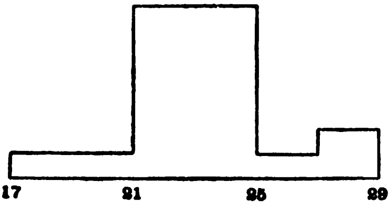

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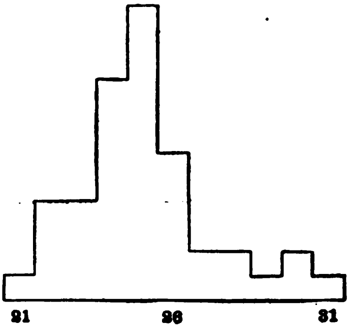

11

FIGS. 4–11.
For descriptions of FIGS. 4–11, see page 32.

Figures 4–11 give each the measurement of some trait in one individual, the traits being as follows:

FIG. 4. Reaction time of *H*, in thousandths of a second; 400 measurements made.

FIG. 5. Quickness of movement of *T*, in seconds; 67 measurements made.

FIG. 6. Quickness in addition of *S*, in seconds; 74 measurements made.

FIG. 7. Number of letters of a certain sort marked on a sheet of mixed letters, by *E* in a given time; 88 measurements made.

FIG. 8. Percentage of men unemployed in the case of a certain trade; 32 years' results measured.[2]

FIG. 9. Attendance of school *E* (the number absent out of 139 enrolled); 74 measurements.

FIG. 10. Daily exchanges of a clearing-house, in $10,000,000s; 19 measurements made.

FIG. 11. Radial pulse of *B*, in seconds required for 30 beats; 44 measurements.

Distributions Vary in Their Geometrical Form.—If it were necessary to pick some one kind of distribution as the best representative of all these, one would choose that approached by Figs. 1, 2, 5, 6, 7. In them we see the separate measures distributed somewhat symmetrically about a single central measure, and decreasing in

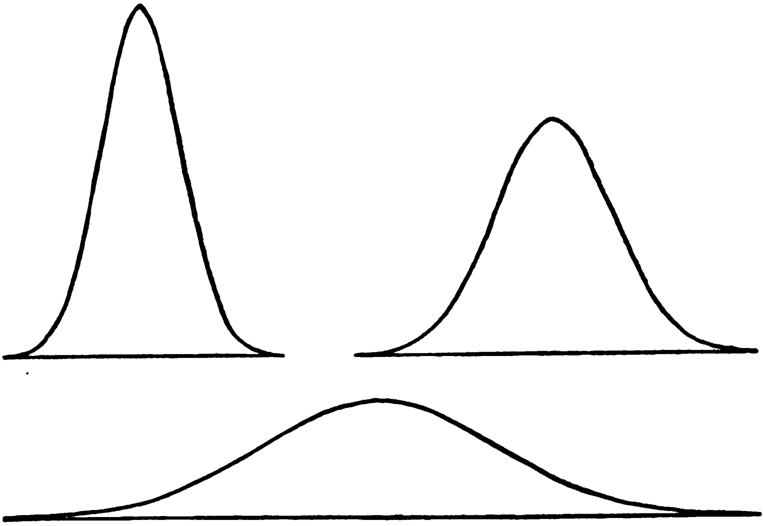

FIG. 12. Type of distribution to which variable traits in individuals often roughly approximate. The three diagrams represent the same geometrical form of surface, the only difference being in the variability.

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[2] Friendly Society of Iron-founders' report, arranged from data given by G. H. Wood, *Journal of the Royal Statistical Society*, Vol. 62, pp. 640–642.

frequency as we pass from the central measure toward either extreme, slowly at first, then more rapidly and then more slowly. They follow roughly the type shown in Fig. 12. But obviously there is no one kind that adequately represents all. The number of central types need not be one, and the variations from the central type may occur in all sorts of ways. Indeed, even in the same trait, there may occur among different individuals different types of distribution. Fig. 13 illustrates this in the case of the accuracy of a certain kind of perceptive process in eleven individuals. The individuals were chosen at random and so give an impartial representation of the fact.

Skewness and Bimodality.—Before discussing further the treatment of a measure expressed in a table of frequencies, it will be well to examine some clearer cases of a hypothetical nature. Suppose, for example, that measures were at hand: (1) of the daily consumption of wealth by an individual, (2*a*) of the hours worked daily by an earnest laborer, whose union did not permit more than an eight-hour day, (2*b*) of the rate of adding of a practised accountant, (3*a*) of the amount of alcohol imbibed daily by a dipsomaniac, and (3*b*) of the number of arrests daily for drunkenness in a city.

An individual who most frequently consumes two dollars' worth in food eaten, clothes worn out, minor luxuries, etc., may consume five dollars' worth by an expensive dinner, ten dollars' worth by burning up his coat, or a hundred dollars' worth by breaking a vase or overdriving a horse. He can not consume less than zero. The range of distribution, limited below, runs out above a long way for practically every one. Letting the scale run from low amounts at the left to high amounts at the right, the form of distribution will be like that of diagram A in Fig. 14, a form *skewed* toward the high end.

The laborer can not work over eight hours, but will less and less readily suffer a greater and greater decrease from that amount due to weather, employer's convenience, etc. The frequency of seven-hour days will be much below that of eight; that of six-hour days below that of seven, etc. I omit from consideration Sundays and holidays. Letting the scale run from 0 at the left up to high amounts at the right, the form of distribution will be like that of diagram B in Fig. 14, being skewed toward the low end. So also

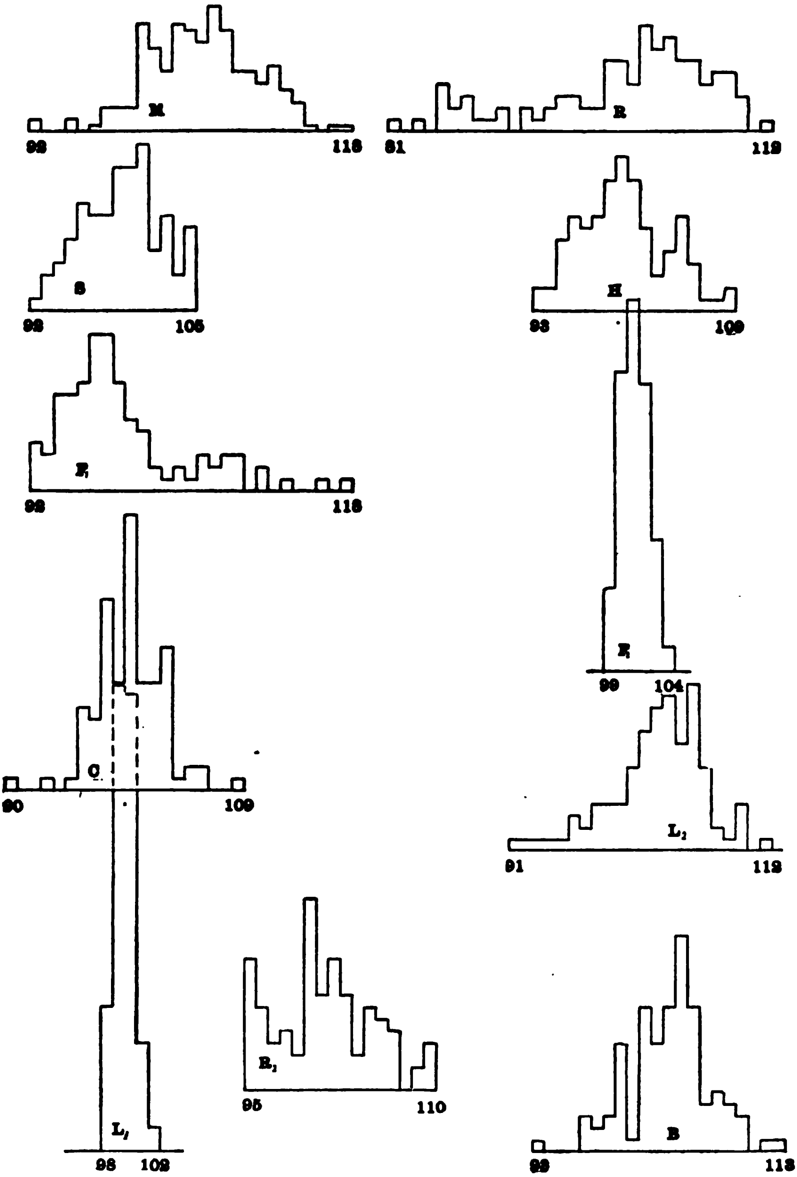

FIG. 13. The surface of frequency for the length of line drawn by a person attempting to equal a 100 mm. line. Eleven individuals are represented—each by one surface of frequency, recording one hundred measurements of his ability in the trait in question. Thus, that for individual *B*, in the lower right hand corner, reads: "The line drawn was 93 (or −7 mm. in error), once; 97 (or − 3 mm. in error), three times; 98 (or − 2 mm. in error), twice; 99 (or − 1 mm. in error), three times; and so on.

the practised accountant will work in most cases near his best rate; but while nothing can raise him far above his customary rate, distraction of attention by outside stimuli, fatigue or bewilderment may drag him far below it.

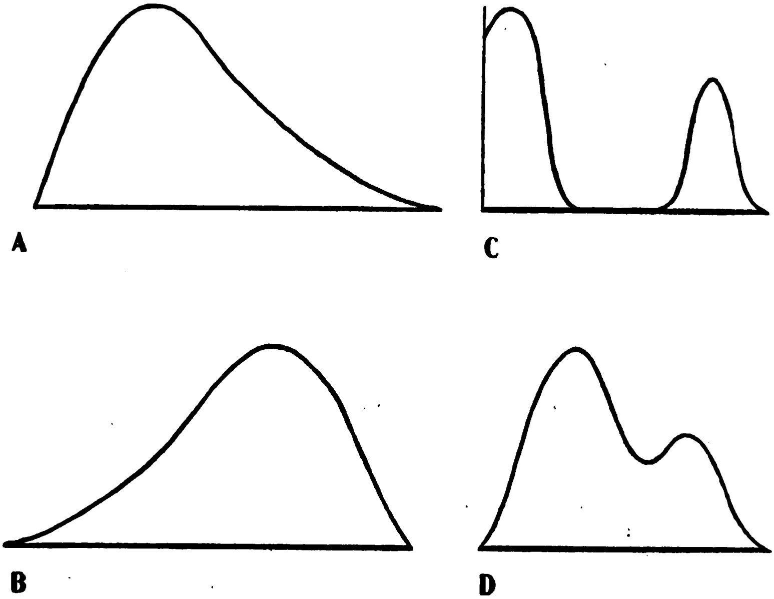

FIG. 14.

The periodic dipsomaniac drinks either a great deal, or little or none, according to the presence or absence of the fit of craving. The distribution of the daily amount of liquor drunk by him will therefore have two points of great frequency, with very slight frequencies for intermediate points, somewhat as shown in diagram C of Fig. 14. The city's daily arrests for drunkenness will show a similar, though not so pronounced, composition of great numbers due to Saturdays, Sundays and holidays, and smaller numbers due to ordinary days. The distribution will verge toward that of diagram D in Fig. 14.[3]

These hypothetical cases emphasize types of clear departure from the common symmetrical, bell-shaped form, and illustrate the importance of giving, to describe a variable fact, all the measures of it that have been made, not simply some one measure that represents their central tendency.

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[3] The scale is, as before, supposed to run from small amounts at the left to larger amounts at the right.

These considerations lead to two simple rules for practise: (1) Many repeated measurements are necessary to measure a variable fact.[4] (2) Turn a series of measures into a distribution table or surface of frequency and examine this before inferring anything from the series.

§ 7. *Measures of Central Tendency*

Nothing short of the entire distribution table is a complete measure of a variable fact, but two features of such a distribution are of special importance: first, the *central tendency* or *typical performance* or amount about which the separate measures cluster, if there is such; and, second, the *variability* or *dispersion* of the separate measures around their central tendency.

Thus in Table 6, *A* and *B* differ in central tendency, but are alike in variability; *C* and *D* differ in variability, but are alike in central tendency.

TABLE 6

HOURLY EARNINGS OF FOUR MEN, *A*, *B*, *C*, AND *D*

Quantity: Cents	Frequency: For *A*	For *B*	For *C*	For *D*
21				2
22				4
23	3		3	8
24	7		7	9
25	20	3	20	15
26	28	7	28	18
27	22	20	22	15
28	7	28	7	11
29	1	22	1	7
30		7		5
31		1		1

Average, Median and Mode.—The *Average*—that is, the sum of a set of measures divided by their number[5]—is the measure of central tendency in most common use. The *Median* or *50 percentile* or *Mid-measure* is the place on the scale reached by counting half of the measures, in the order of their magnitude, or the place on the scale above and below which are equal numbers of the measures. Thus in the series 5, 6, 7, 20, 22 the median is 7. An amount which appears more frequently than the other amounts on either side of it in the distribution of the measures of a variable fact may be called a *Mode*, or better, a *Crude Mode*. Thus in Table 4 (page 29) 100, or 0 error, is the crude mode, and in Table 5, 1–2 and 6–7 are the crude modes. In Table 6 (page 36) 26 cents is the mode for both *C* and *D*, and also for *A*.

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[4] The number needed will be discussed in Chapter XII.

[5] *Average* will be used throughout this book for the *arithmetical* mean or average, unless special notice is given to the contrary.

The arithmetical average is often unwisely used as the sole measure of central tendency. But it is clear that the average of the man's daily consumption of wealth figured in Fig. 14 *A* not only does not distinguish him from some one less given to extreme prodigality who in general lives on a higher material plane, but also gives no idea of his common daily expenses. So also the average performance of an accountant (see Fig. 14, *B*) may not tell what is really desired, namely, what the man can do under proper conditions. With a case like that of the dipsomaniac (see Fig. 14, *C*) the average grossly misrepresents the facts to all readers who follow the common habit of expecting an average to be approximately the individual's typical performance. An average is mathematically only the sum of a set of measures divided by their number. It represents the typical measure of the set only when there is but one typical measure and when the set of measures are symmetrically disposed about it. There may be more than one type of measure prominent, and the distribution may be, and often is, skewed instead of symmetrical.

Central Tendencies in Unimodal Distributions.—When the different measures of a variable fact cluster around one and only one such point of notable frequency or typical amount of the fact (as in Figs. 5, 6, 7 on page 31, or Fig. 12 on page 32, or Fig. 14, *A* and *B*, on page 35), their distribution is called *unimodal*. When they show two or more such modes (as in Fig. 14, *C* and *D*, on page 35), their distribution is called *multimodal*.

In a unimodal distribution the different measures may be distributed along the scale symmetrically with respect to the mode or place of greatest frequency, or approximately so, so that the two portions of the surface of frequency on either side of a vertical erected at the mode are approximately alike; or the different measures above the mode may be distributed differently from those below it, so that the two portions of the surface of frequency differ, the distribution being *skewed.* Perfectly symmetrical distributions are shown in Fig. 12 (page 32) and in Figs. 17 and 18 (page 66). Fig. 7 (page 31) is approximately symmetrical. Figs. 4 and 8 (page 31), Fig. 14, *A* and *B* (page 35), and Figs. 19 and 20 (page 68) show clear skewness.

In a distribution that is unimodal and symmetrical the *average* is identical with the *mode* and represents the typical performance of the individual. The *50 percentile* or *median* (that is, the point on the scale—or the amount of the trait—above which and below which are equal numbers of the different measures) will also be identical with the average and the mode. In a distribution that is approximately unimodal and approximately symmetrical, the average or the mode or the median will represent approximately the central tendency about which all the varying measures cluster. Thus, in the case of the eighty-eight hours' earnings of *C*, recorded in Table 6, the average is a trifle under 26, the median is 26 and the mode is 26. In a distribution that is unimodal and skewed the mode often gives much more useful information than the median or average. In the case of skewed distributions it is specially important to bear in mind the meaning of whichever means of representing central tendency is used. The average tells the general weight of the fact, the mode tells its usual or "typical" amount and the median, or 50 percentile, gives a mongrel result, often useful just because it pretends to be neither the type nor the general weight of the fact, but only a certain unambiguous feature of it.

The following further characteristics of the different measures of central tendency may help to decide which is the best to use in any given case:

The crude mode is the most easily and quickly determined. It is not so reliable a measure as the others. That is, the actual mode obtained from a given number of cases will not be so near the true mode as will the actual average to the true average. It is hardly at all influenced by extreme measures or erroneous measures. It is unambiguous and does not mislead a reader into thinking that all the individual measures of a group are very closely near it.

The median is more easily determined than the average. It is not so precise as the average, is very little influenced by extreme or erroneous measurements and is unambiguous.

The average is determined more precisely than the crude mode or the median because the amount of every measure plays a part in determining it, but for this very reason it is more influenced by extreme or erroneous measures. The average is the measure in common use and has the advantage of being a familiar term, and at the same time the disadvantage of leading untrained readers to think that the abilities of which it is the average are closely clustered about it.

In the case of skewed distributions the crude mode has the obvious special advantage of being closest to the "typical" amount of the trait.

If the measures of an individual are not in terms of amount, but are simply a series ranked in relative position, the only measures of central tendency available are the mode and median.

Central Tendencies in Multimodal Distributions.—A multimodal distribution almost always means that two facts have been measured that need to be kept separate in thought. So the different modes should be kept separate, and, if possible, the total distribution should be analyzed into separate distributions, whose central tendencies are then treated separately.

§ 8. *Measures of Variability*

Measures of the variability or dispersion of the individual measures are of two sorts. There are measures obtained by averaging the deviations of the individual measures from their central measure, and measures of the limits which include a certain proportion of all the individual measures.

Measures by Averaging.—Of the first sort we have the *Average Deviation* or Mean Variation (A.D. or M.V.), which equals the average of the deviations (all treated as positive quantities) of the individual measures from their central tendency (average, median or mode); and the *Mean Square Deviation*[6] (S.D. or σ) which equals the square root of the average of the squares of the deviations of the individual measures from their central tendency. Thus, calling the series of measures $m_1, m_2, \ldots, m_n$ and their deviations from the measure taken as central tendency $x_1, x_2, \ldots, x_n$, A.D. $= \Sigma x/n$ and S.D. $= \sqrt{\Sigma x^2/n}$, Σ being a symbol for 'the sum of the' and n equalling the number of measures. The m's being, for example, 8, 9, 10, 10, 11, 12, 12, 12, 12, 13, 13, 14, 15, 17, the x's are -4 -3, -2, -2, -1, 0, 0, 0, 0, $+1$, $+1$, $+2$, $+3$, $+5$; the A.D. is 24/14 or 1.7; the x^2s are 16, 9, 4, 4, 1, 0, 0, 0, 0, 1, 1, 4, 9, 25; the S.D. is $\sqrt{74/14}$ or 2.3.

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[6] The mean square deviation is sometimes called the *Standard Deviation*, though its right to be considered a standard measure of variability is by no means secure. S.D., σ, or Mean Square Deviation will be used indifferently to refer to it throughout this book.

Measures by Limits Required to Include a Specified Percentage of the Cases.—There are many possible measures of the second sort. For example, in the case just used, all the measures are between 7½ and 17½, 86 per cent. are between 8½ and 16, 71 per cent. are between 9½ and 14½, 50 per cent. are between 10½ and 13½. Two measures of this sort are in common use. One is the *Q*, or 'Semi-Interquartile-Range,' or '*half the distance between the 25 percentile measure* (here 10) *and the 75 percentile measure* (here 13).' It is 1½ in the present case. The other is the *Med. Dev.* or P.E.,[7] the *median of the deviations from the central tendency* (all being considered as positive quantities). In the present case, this is 1½, since there are four deviations of 0, three of 1, and seven of 2 or more.

Such measures by limits have the advantages of economy of time in calculation and of being only slightly influenced by extreme or erroneous measures. They are the only measures of variability available when the measures are of relative position. They have the disadvantage of being less precisely determined (the same data being given) than the A.D. or S.D.

If the distribution is symmetrical, an A.D., S.D., Med. Dev. or the like suffices to summarize the variability or dispersion of the separate measures about their central tendency. If the distribution is skewed the variability above and that below the central tendency need to be measured separately.

Variability in Multimodal Distributions.—If the distribution is multimodal it should be analyzed into separate distributions and the variability of each should be measured separately.

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[7] The P.E. stands for Probable Error, the traditional, but very misleading, name for the median deviation, which is not specially probable and not an error at all.

PROBLEMS

8. Express in tables of frequencies and surfaces of frequency the following facts:

Ar., being measured with respect to his memory span for letters 40 times, showed the following abilities, in terms of the number of words remembered in their correct positions: 7, 6, 7, 5, 8, 2, 10, 6, 7, 8, 3, 8, 6, 9, 6, 10, 6, 8, 6, 4, 9, 6, 10, 8, 6, 8, 5, 6, 4, 8, 10, 7, 4, 7, 6, 9, 1, 11, 7, 7.

D., being measured in the same trait 40 times, showed records of: 5, 4, 1, 6, 5, 5, 8, 4, 6, 5, 5, 5, 4, 6, 4, 4, 5, 7, 2, 5, 5, 4, 5, 4, 6, 9, 4, 3, 0, 5, 5, 6, 5, 6, 3, 8, 4, 5, 5, 3.

9. Judge by inspection which is the more variable, Ar. or D.?

10. In which case is it most clearly a matter of indifference whether the general tendency is expressed by the average or by the median or by the mode?

11. Find the Crude Mode, the Median, the Average, the Average Deviation (A.D.) from the Median, the S.D. from the Median, the Median Deviation (P.E.) from the Median and the Q, in the case of each of these two series of measures:

SERIES I.	16,	24,	26,	23,	23,	21,	22,	11,	18,	20,
	19,	25,	19,	24,	22,	23,	23,	21,	22,	18.
SERIES II.	11,	21,	21,	22,	19,	23,	20,	22,	23,	23,
	21,	24,	22,	24,	22,	20,	22,	24,	26,	23.

12a. What are the closest limits which will include 75 per cent. of Series I.?

12b. What are the closest limits which will include 75 per cent. of Series II.?

CHAPTER IV

THE ARITHMETIC OF CALCULATING CENTRAL TENDENCIES AND VARIABILITIES[1]

§ 9. *Calculations from Measures Taken at Their Face Value*

Consistency in Units.—The arithmetic of calculating averages, medians, modes, quartiles, A.D.'s, S.D.'s, P.E.'s and other measures of central tendency and of variability from a series of measures taken at their face value is simple and straightforward, if one bears in mind (1) that mental and social quantities are commonly continuous, so that any number given as a measure means not a point, but a distance on the scale, and (2) that this distance is often that from the given number to the next number, so that the real value of the number is itself plus one half of the difference between it and the next number.

Thus, in measurements of a quantity that varies continually, 8 will mean either from 7.5 to 8.5 or from 8.0 up to 9.0; 8.2 will mean from 8.15 to 8.25 or from 8.20 up to 8.30; 8.27 will mean from 8.265 to 8.275 or from 8.270 up to 8.280—in each case according to the fineness of the scaling and according to whether the persons obtaining the data measured 'to the nearest number of the scale' or 'between two neighboring numbers of the scale.' 7, 8, etc., are sometimes used carelessly for 7.0 and 8.0 or 7.00 and 8.00. The real space on the scale meant by the number is usually evident in such cases from inspection of the series as a whole.

Short Methods.—Some of the calculations, though simple, are very tedious unless short methods are used. Command of these methods is indeed essential for anyone who is to use his time intelligently in quantitative work. Since they are in some respects foreign to the mathematical habits of one's school days they require comment and illustration. It is also well to make acquaintance with certain justifiable methods of attaining approximate results in cases where the worker's time is worth more to science than slightly increased precision.

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[1] This chapter concerns only unimodal distributions and multimodal distributions whose modes are not very pronounced—distributions, that is, which may fairly be considered in each case as varying around a typical condition.

Definitions Restated and Illustrated.—Before describing and illustrating the technique of the simpler short methods and approximations I will repeat the definitions of the measures to be calculated and illustrate them in the case of the following series of measures: 9, 10, 12, 13, 13, 14, 14, 14, 15, 15, 15, 15, 16, 16, 17, 17, 17, 18—the quantity measured being a continuous variable.

The scale runs from 9 to 18 by steps of 1. We will suppose the measurements to have been taken to the nearest point on the scale.

The number of measures, $n = 18$.

The Crude Mode is the most frequent measure, here 15.

The Median or 50 percentile is the measure above and below which are equal numbers of the measures,[2] here 15.

The Average is the sum of the measures divided by their number, $\Sigma m/n$, here 14.44.

The Average Deviation, A.D., is the average of the differences (regardless of signs) between the separate measures and their central tendency.

The A.D. from the mode is here 32/18, or 1.78.

The A.D. from the median is here the same, 1.78.

The A.D. from the average is here 33.1/18, or 1.84.

The Mean Square Deviation, S.D. or σ, is the square root of the average of the squares of the differences between the separate measures and their central tendency.

The S.D. from the mode is here $\sqrt{104/18}$, or 2.4.

The S.D. from the median is here the same, 2.4.

The S.D. from the average is here $\sqrt{98.4/18}$, or 2.34.

The 25 percentile is the measure with three times as many measures above as below it,[3] here 13.

The 75 percentile is the measure leaving one third as many measures above as below it,[4] here 16.

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[2] Or the measure reached, as the $(n + 1)/2$th, in counting the measures in the order of their magnitude.

[3] Or the measure reached by counting one fourth of the measures in the order of their magnitude beginning with the lowest.

[4] Or the measure reached by counting three fourths of the measures in the order of their magnitude from the lowest.

The Quartile, or Q, is half the difference between the 25 percentile and the 75 percentile, here 1.5.

The Median Deviation or P.E. is the median of the differences (regardless of signs) between the separate measures and their central tendency.

The Med. Dev. or P.E. from the Mode is here 1.5, four of the differences being 0; five, 1; and the other nine, 2 or more.

The Med. Dev. or P.E. from the median is likewise 1.5.

The Med. Dev. or P.E. from the average is 1.5 (between 1.44 and 1.56), three of the differences being .44; four, .56; two, 1.44; and the other nine, 1.56 or more.

If the measure 9 meant 9.0 up to 10.0, the measure 10, 10.0 up to 11.0, etc., the measures of central tendency, and also the 25 percentile and 75 percentile, would each be raised by .5. The A.D., S.D., and Med. Dev. or P.E. would be unaltered.

Short Methods in General.—To save eye-strain and reduce errors, use paper ruled into squares of about ⅛ inch by light blue lines for all computations. In all cases arrange the measures as a table of frequency, scale values to the left. Treat one 'step' of the scale as 1, no matter what its real value is, reducing answers back to the real value. Thus suppose the distribution to be:

Quantity: Dollars	Frequency
110 to 118	2
118 to 126	4
126 to 134	7
134 to 142	9
142 to 150	6
150 to 158	3
158 to 166	1

and the problem to be: To get the A.D. from the Crude Mode. The Crude Mode is at the 134–142 step. There are then nine deviations of 0 steps, 13 (7 + 6) of 1 step, 7 (4 + 3) of 2 steps, and 3 (2 + 1) of 3 steps. The A.D. is then 36/32 steps, or 1.125 steps. But each step is $8, so that the A.D. is $9.

Calculation of the Average.—The labor of calculating averages can be much reduced by adopting the method which most of us would probably use in a case like this: To get the average of 54, 52, 64, 56 and 50. Remembering that the average is such a figure that the sum of differences between it and the measures above it is equal to the sum of the differences between it and the measures below it, one takes 56 as a guess. The differences below are 2, 4 and 6, that above is 8. If the average was altered by — .8, or to 55.2, the differences below would be 1.2, 3.2 and 5.2, and those above would be 8.8 and .8. This common procedure consists in guessing at an approximate average and then correcting it from knowledge of the sums of the minus and plus deviations from it. It lets us add small numbers instead of large and, as will be seen, gives us at the same time as the average, an approximate measure of the average deviation from it.

The choice of an approximate average is commonly easy after an inspection of the total distribution, and one soon acquires skill in making a correct choice in any case.

Suppose the measures to be as follows:

REACTION-TIMES OF V. H.

Quantity: Seconds	Frequency	
.120 to .12499 or .1225	2	-
.125 to .12999 or .1275	3	- 6
.130 to .13499 or .1325	11	- 5
.135 etc.	13	- 4
.140	11	- 3
.145	13	- 2
.150	7	- 1
.155	8	— 0
.160	13	+ 1
.165	8	+ 2
.170	1	+
.175	3	
.180	3	+
.185	0	
.190	0	
.195 to 19999	1	

In this and all following calculations from measures taken at their face value, any unit distance on the scale (that is, the "interval" or "step") is to be represented by its mid-point. That is, of course, its face value.

Either .1450–.1499 or .1500–.1549 would do for a guess. I will use .1450–.1499. We have then to obtain the minus and plus differences of all the separate measures from the ".1450–.1499" step. To save labor in multiplication and addition we measure these in terms, not of units of the scale, but of steps of the scale—*i. e.*, using five thousandths of a second as unity. We have then 13 deviations of 0 and, for minus and plus deviations:

2 deviations of −5 or −10
3 deviations of −4 or −12
11 deviations of −3 or −33
13 deviations of −2 or −26
11 deviations of −1 or −11
40 −92

7 deviations of + 1 or + 7
8 deviations of + 2 or + 16
13 deviations of + 3 or + 39
8 deviations of + 4 or + 32
1 deviations of + 5 or + 5
3 deviations of + 6 or + 18
3 deviations of + 7 or + 21
0 deviations of + 8
0 deviations of + 9
1 deviations of +10 or + 10
44 +148

The approximate average is evidently too low. It can be corrected by adding to it the algebraic sum of the deviations divided by the number of cases. In the illustration this will be $+\frac{56}{97}$ or + .58. .58 of a step = 2.9 thousandths of a second. The corrected average is then the mid-point of the '.145 to .1499' step + .0029, or .1475 + .0029, or .1504 sec. Calling the algebraic sum of the deviations from the approximate average divided by the number of cases $d_{\text{act. av.—approx. av.}}$, or simply d, and calling the approximate, or guessed, average G.A., we may summarize this whole calculation in the formulæ:

$$\text{Av.} = \text{G.A.} + \text{d},$$
$$\text{d} = (\Sigma \text{ dev. (alg.) from G.A.})/\text{n}$$

Calculation of the Average Deviation from the Average.—The procedure here is to use the sum of the deviations of the separate measures from the approximate average (G.A.), correcting it to what it would be had the deviations been reckoned from the actual average. The procedure is simple. In the illustration the sum of the deviations (all treated as positive quantities) from the G.A. was 240 (*i. e.*, 92 + 148), the unit being one step of the scale. Since the actual average is .58 step higher than the approximate average, 53 of the separate deviations from the G.A. (13 zero and 40 minus deviations) will be increased, each by .58, when measured from the Av., and 44 of them will be decreased, each by .58. The sum of the deviations (regardless of signs) from the Av. will then be 240

+ (53 × .58) − (44 × .58), or 245.22. The A.D. from the Av. will then be 245.22/97 or 2.528 (still in units of a step), or .0126 second.

This procedure may be generalized as follows: for cases where d < 1 "step"

Call the separate measures the *m*'s.

Call the total number of measures *n*.

Call the approximate or guessed average G.A., and the actual average Av.

Call Av. − G.A., *d*. Let *d* be given its algebraic value, + or −.

Call the sum of the deviations of the separate measures, regardless of signs, from G.A., $\Sigma\xi$.

Call the sum of the deviations of the separate measures (again regardless of signs) from Av., Σx.

Call the number of *m*'s which are less than the Av., *l*.

Call the number of *m*'s which are greater than the Av., $n - l$.

Then, further use of signs being algebraic,

$$\Sigma x = \Sigma\xi + [l.d] - [(n - l)d],$$

and

$$\text{A.D. from Av.} = \frac{\Sigma\xi + [l.d] - [(n - l)d]}{n}$$

The A.D. from the Median. The A.D. from the Mode.—The procedure in calculating the average deviation from the median or from the crude mode is simply that used in calculating the average deviation from the general average, the median and crude mode being, by the face-value or mid-point method, always at the mid-point of some step of the scale.

Calculation of the Mean Square Deviation (S.D. or σ) from the Average.—Find the sum of the squares of the deviations of the separate measures from the approximate average. Call this sum, $\Sigma\xi^2$. Infer the S.D. from it as shown below. Since the deviations themselves have been computed in the calculation of the average, the sum of their squares is obtainable by easy multiplication, by 1, 2, 3, 4, with − or + signs as the case may be, etc. Thus in our illustration we have already the first column below.

Minus Deviations	Multiplier	Squares of Deviations
−10	−5	50
−12	−4	48
−33	−3	99
−26	−2	52
−11	−1	11
−92		260

Plus Deviations		
7	1	7
16	2	32
39	3	117
32	4	128
5	5	25
18	6	108
21	7	147
—		—
—		—
10	10	100
148		664

$\sqrt{\frac{\sum \xi^2}{n} - d^2}$

$\Sigma\xi^2$, the sum of the square of the deviations of the *m*'s from G.A. = 260 + 664, or 924. The S.D. from the actual average = $\sqrt{\frac{\Sigma\xi^2}{n} - d^2}$,[5] d equaling, as before, Av. − G.A. S.D. from the actual average therefore = $\sqrt{\frac{924}{97} - (.58)^2}$, or 3.03 (in units of one step), or .015 sec.

The Calculation of Percentile Values.—The 25 percentile, 50 percentile or median, 75 percentile, and Q for a series of measures taken at their face value can be obtained most conveniently by writing down the necessary sums of the *numbers* of the measures (not of their amounts) from the beginning. Thus, all that is necessary in our illustration is to list sums beside the column of frequencies and do the simple computations, as shown below.

Quantity: Thousandths of a Sec.	Frequencies	Sums from the Beginning
120–124.99	2	2
125–	3	5
130–	11	16
135–	13	29
140–	11	40
145	13	53
150	7	60
155	8	68
160	13	81
165	8	89
170	1	90
175	3	93
180	3	96
185	0	
190	0	
195–199.99	1	97

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[5] The student may take this formula on trust; or verify it empirically; or, if possessed of the requisite knowledge of algebra, deduce it.

$$97/4 = 24.25, \quad 97/2 = 48.5, \quad 3/4 \text{ of } 97 = 72.75.$$

The 25 percentile is obviously in step *135 up to 140* or, using its mid-point, at 137.5 of the scale, or .1375 sec.

The median percentile is obviously in step *145 up to 150* or, using its mid-point, at 147.5 of the scale, or .1475 sec.

The 75 percentile is obviously in step *160 up to 165* or, using its mid-point, at 162.5 of the scale, or .1625 sec.

The Q is then $\dfrac{.1625 \text{ sec.} - .1375 \text{ sec.}}{2}$ or .0125 sec.

The Calculation of the Median Deviation from the Average.—The Median Deviation or P.E. from the average is obtained either by relisting the measures in the order of their amount of deviation from the average, as shown below for our illustration, or by simple inspection.

	m's		*m*'s	Sums of *m*'s from the Beginning
+ .42 step	7			7
		− .58 step	13	20
+1.42 "	8			28
		−1.58 "	11	39
+2.42 "	13			52
		−2.58 "	13	etc.
+3.42 "	8			
etc.		−3.58 "	11	

The median of the deviations is thus at 2.42 (in units of a step), or .0121 sec.

The Median Deviation from the crude mode is got similarly, but more easily, since the deviations are all in integral multiples of a step. The Median Deviation from the Median is the same as Q.

Approximations. Grouping.—Time can be saved by grouping measures in a distribution more coarsely than by their face value. Thus the series of our illustration may be distributed by steps of ten thousandths of a second instead of by the steps of five in which the measures at their face value appeared. We then have:

Quantity: Seconds			Frequency
.120	up to	.130	5
.130	"	.140	24
.140	"	.150	24
150	"	.160	15
160	"	.170	21
170	"	.180	4
180	"	.190	3
190	"	.200	1

In general, in mental and social measurements, in the calculation of averages, average deviations and mean square deviations, when the face value of the series gives a grouping of 40 to 60 steps, it is allowable to group by double steps, and, when the face value of the series gives a grouping of 60–80 steps, to group by triple steps. But it should be observed that coarse grouping saves little time except in the calculation of the average, average deviation and mean square deviation. In the case of the calculation of the median, 25 percentile, 75 percentile, and median deviation, it is the author's opinion that the gain in precision from the finer scale is greater than the loss in time, if one economizes time in recording the measures in the finer grouping by some such method as the following:

Suppose the measures to be: 411, 432, 444, 451, 456, 463, 471, 477, 480, 484, 492, 495, 495, 501, 507, 512, 513, 516, 519, 525, 527, 532, 533, 544, 552, 566. The range of the distribution is 155 units. All can, however, be recorded easily thus:

41	1
42	
43	2
44	4
45	1, 6
46	3
47	1, 7
48	0, 4
49	2, 5, 5
50	1, 7
51	2, 3, 6, 9
52	5, 7
53	2, 3
54	4
55	2
56	6

In getting the Av., A.D. and S.D., the series can easily be treated as one of 16 steps of 10, but in getting the Med., Med. Dev., *Q* or other percentile values, advantage can be taken of the full detail.

Approximations for the Median Deviation or P.E.—Time can be saved in calculating the Med. Dev. or P.E. by using *Q* as an approximation to it. If the distribution is symmetrical, *Q* has the same value as the median deviation, and if the distribution is not symmetrical *Q* approximates to, and is at least as useful a measure of variability as, the median deviation.

§ 10. *Calculations of Values More Probable than those Got from Measures Taken at their Face Value*

So far computations have been considered on the principle that any measure should be taken at its face value, the face value of a measure in the case of a measure covering a certain distance on a scale being the mid-point of that space.

It is possible in the case of continuous quantities[6] to estimate central tendencies and variabilities more precisely by considering certain other possibilities for the treatment of a measure like 19.4 (measuring from 19.35 up to 19.45) than to replace it by its mid-point, 19.40000 etc. For example, consider the calculation of the 50 percentile or median in our illustrative case whose data for convenience I repeat below.

Quantity			Frequency
.120	up to	.125	2
.125	"	.130	3
.130	"	.135	11
.135	"	.140	13
.140	"	.145	11
.145	"	.150	13
.150	"	.155	7
.155	"	.160	8
.160	"	.165	13
.165	"	.170	8
.170	"	.175	1
.175	"	.180	3
.180	"	.185	3
.185	"	.190	0
.190	"	.195	0
.195	"	.200	1

$n = 97$

Inferences from Continuity.—The number of measures being 97 we have to count in 48½ measures. 40 measures reach to the end of the ".140 up to .145" step. 48½ measures will then reach to the ninth measure of the 13 which are in step "*.145 up to .150.*" What is absolutely known is that the median is somewhere from

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[6] It should be noted that the principles in Sections 10 and 11 apply not only to surfaces of frequency of truly continuous variables such as time, legibility of handwriting, value in exchange, amount of zeal, stature, knowledge of German and the like, but also to surfaces of frequency of discrete variables, *when the "steps " in which the measures are reported include each more than one of the ultimate discrete steps* by less than which the actual fact cannot vary. Call all such cases

.145 up to .150. By the face-value methods we locate it at the point .1475, which stands for the measure "*.145 up to .150.*" But suppose that the measures had been reported to a ten-thousandth of a second. It is not likely that 8 of the thirteen would have been below .1475 and only 4 above it. The median would, with fine enough scaling of the data, probably have been nearer .150 than .145, since it should be the point on the scale between .145 and .150 leaving 8½ of the 13 measures below it. If, instead of merely accepting the mid-value for all the '*.145 up to .150*' measures, we use probabilities to estimate how the thirteen measures would be spread from .145 to .150 we may make a probable estimate of the median for the distribution more precisely than as "somewhere in the .145–.150 space, call it at the mid-point." For example, an estimate of the median as .145 + 8½/13 of the step—that is, as .1483 sec.—is probably truer than .1475 sec.

Consider further the calculation of the median in this same distribution, but arranged with a coarser grouping, as follows:

Quantity	Frequency	Sums from Beginning
.120 up to .130	5	5
.130 " .140	24	29
.140 " .150	24	53
.150 " .160	15	68
.160 " .170	21	
.170 " .180	4	
.180 " .190	3	
.190 " .200	1	

pseudo-continuous variables. For example, if the enrollment of a school is measured as:

Quantity: Number of Pupils Enrolled	Frequency: Number of Daily Registers
410–419	1
420–429	3
430–439	7
440–449	12
450–459	16
460–469	22
470–479	27
480–489	33
490–499	28
500–509	22
510–519	17
520–529	2

The fact, as reported, is *pseudo-continuous* and the treatment will be that of a continuous variable.

Counting in 48½ measures, we locate the median in the "*.140–.150*" step and, if we replaced this step by its mid-point, should call the median .145. But if we used probabilities in placing the point to be reached by counting 19½ of the 24 cases in the "*.140 up to .150*" step arranged in order of magnitude, putting it at 19½/24 of the distance from .140 to .150, that is, at .1481, we should obviously come nearer the truth as shown by the finer grouping.

Inferences from the Form of the Distribution.—Consider finally the calculation of the 75 percentile from this coarser grouping. It is required to count in 72¾ cases from the low extreme. This brings us into the "*.160 up to .170*" step, 68 cases being below .160. Now .165, the mid-point of the '*.160–.170*' group, would be an inferior estimate for the 75 percentile not only because only 4¾ out of 21 cases need to be taken but also because the general slope of the distribution thereabouts (24, 21, 15, 4, 3, 1) shows that probably the cases would be much more frequent, if reported with a fine grouping, toward .160 than toward .170.

There are then two sorts of facts that may help in estimating central tendencies and variabilities with a greater probable exactitude than is secured by treating each step of the scale arithmetically as its mid-point. First, the cases located within that step may be thought of as spread over it, and, second, the general form of the distribution may be used as a means of judging *how* they will probably be spread. We may, that is, calculate central tendencies and variabilities with the aid of estimates of how the separate measures within each step would be spread over it if the scaling had been very, very much finer.

The value of using these probabilities to refine our estimate of central tendencies and variabilities depends evidently on the fineness of the grouping of the measurements as they are. The coarser the grouping, the more desirable it becomes to consider these two lines of facts.

The following notes give the technical procedure desirable in calculating probable central tendencies and variabilities with the aid of an estimated spread of the cases over the distance denoted by any step.

The Average.—No change from the face value or mid-point method is necessary.

The Median.—The cases within the step where the median lies may be considered as spread evenly[7] over the whole distance of the step, and the median point placed accordingly. Thus in our illustrations, since 48½ cases reach to the end of the 8½ cases of the 13 between .145 and .150 the median point may be placed at $.145 + \frac{8.5}{13}(.005)$.

The Mode.—The question here is troublesome. By the definition of the crude mode as given, "the most frequent measure," the crude mode in the case of a continuous quantity is not a point but a distance. To replace this distance by any point, whether the mid-point or one more suitable, is not so necessary as with the average or median, since the mode is used oftener to describe a type than as a single number to compute with. It may, however, be necessary for the calculation of variabilities from the mode and for exact comparison with other facts.

The student may use common sense in picking a point to represent the mode or he may follow certain fixed rules, which are however valid only for certain sorts of distributions, so that in the end common sense must decide whether to follow them, or he may make elaborate calculations of the location, on the scale, of the probable point of greatest frequency of the fact.

The use of common sense consists chiefly in observing the neighboring frequencies, so as to pick a point which they make probable. The best fixed rule for general practise is:

Mode = Av. − 3(Av. − Med.) (calculation being algebraic).

The A.D.—The average deviation calculated by the mid-point method tends to be too large since, with fine grouping, more than half the cases within any step would as a rule be within the half of that step toward the central tendency. This may be corrected for as follows:

---

[7] This is not exactly true for any distribution, save by chance, but the difficulties of estimating just how the cases would, for any given distribution, be spread, would be very great and the resulting greater precision trifling.

[8] Let *dis.* stand for *distribution* here and later.

Approximate Correction for Coarse Grouping—A.D.

To estimate the probable A.D. from a very, very fine grouping:

If the dis.[8] ranges over 20 steps or more the correction is negligible
If the dis. ranges from 15 to 20 steps subtract .005 step
If the dis. ranges over 14 steps subtract .005 step
If the dis. ranges over 13 steps subtract .005 step
If the dis. ranges over 12 steps subtract .01 step
If the dis. ranges over 11 steps subtract .01 step
If the dis. ranges over 10 steps subtract .01 step
If the dis. ranges over 9 steps subtract .015 step
If the dis. ranges over 8 steps subtract .015 step
If the dis. ranges over 7 steps subtract .02 step
If the dis. ranges over 6 steps subtract .02 step

If this correction is to be made, it is still easier to get an approximate A.D. from whatever mid-point of a step is nearest the actual average, counting the measures within that step as all deviating by zero, and then to *add*, according to the coarseness of grouping, as follows:

If the dis. ranges from 20 to 50 steps add to A.D. app. .01 step
If the dis. ranges from 15 to 19 steps add to A.D. app. .015 step
If the dis. ranges from 10 to 14 steps add to A.D. app. .02 step
If the dis. ranges over 9 steps add to A.D. app. .025 step
If the dis. ranges over 8 steps add to A.D. app. .03 step
If the dis. ranges over 7 steps add to A.D. app. .035 step
If the dis. ranges over 6 steps add to A.D. app. .04 step
If the dis. ranges over 5 steps add to A.D. app. .05 step

The S.D.—The mean square deviation as obtained by the face-value or mid-point method may be given a probably more accurate value by correction as follows:

Approximate Correction for Coarse Grouping—S.D.

To estimate the probable S.D. that would be got from a very, very fine grouping:

If the dis. ranges over 40 steps or more the correction is <.001 step
If the dis. ranges from 30 to 39 steps subtract from the obtained S.D. .001 step
If the dis. ranges from 25 to 29 steps subtract from the obtained S.D. .002 step
If the dis. ranges from 20 to 25 steps subtract from the obtained S.D. .003 step
If the dis. ranges from 15 to 20 steps subtract from the obtained S.D. .005 step
If the dis. ranges over 14 steps subtract from the obtained S.D. .007 step
If the dis. ranges over 13 steps subtract from the obtained S.D. .008 step
If the dis. ranges over 12 steps subtract from the obtained S.D. .01 step
If the dis. ranges over 11 steps subtract from the obtained S.D. .01 step
If the dis. ranges over 10 steps subtract from the obtained S.D. .01 step
If the dis. ranges over 9 steps subtract from the obtained S.D. .02 step

If the dis. ranges over	8	steps subtract from the obtained S.D.	.025 step
If the dis. ranges over	7	steps subtract from the obtained S.D.	.03 step
If the dis. ranges over	6	steps subtract from the obtained S.D.	.04 step

It will be observed that with a fairly fine scaling, resulting in 20 or more steps in the distribution's range, the S.D. is, to the second decimal, the same as it would be, with very fine grouping. It is customary to make a less complete correction by the formula[9]

$$\text{S.D.} = \sqrt{(\text{S.D.}_{\text{mid.}})^2 - \frac{1}{12}}$$

in which $\text{S.D.}_{\text{mid.}}$ = S.D. calculated by the face-value or mid-point method.

The 25 Percentile, 75 Percentile and Other Percentile Values.—In such unimodal distributions as are in question, the cases within the scale-interval or step wherein the 25 percentile lies will probably be fewer toward the extreme of the distribution than toward the median. Similarly for the cases within the step wherein the 75 percentile lies.

If the student does not take account of the slope of the frequency curve but simply treats the cases within each step or interval as spread evenly over that step, he will probably improve his estimates over what they would be by the face-value or mid-point method. If he does wish to take account of the slope, the most convenient way to do so is by spreading the n_k cases over the interval, "a to $a + k$," putting for each successive tenth of k the fraction of n_k which is appropriate in view of the general slope.

As practical rules the following will lead to adequate precision for any work which the student is likely to have to do.

If the cases are so grouped as to have a range of thirty or more intervals or steps of the scale, treat the cases within one interval as spread evenly over it.

If the grouping is coarser, consider the n_k cases of the interval, 'a to $a + k$' within which the 25 percentile lies, as if they were distributed as follows:

a to $a + .1k$ or	0 to .1 step	9 per cent. of n_k.
$a + .1k$ to $a + .2k$ or	.1 to .2 step	9 per cent. of n_k.
$a + .2k$ to $a + .3k$ or	.2 to .3 step	9 per cent. of n_k.
etc.	.3 to .4 step	10 per cent. of n_k.

---

[9] Sheppard's formula.

.4 to .5 step	10 per cent. of n_k.
.5 to .6 step	10 per cent. of n_k.
.6 to .7 step	10 per cent. of n_k.
.7 to .8 step	11 per cent. of n_k.
.8 to .9 step	11 per cent. of n_k.
.9 to 1.0 step	11 per cent. of n_k.

Or, more exactly, in dependence on the coarseness of grouping, consider the cases within the step where the 25 percentile lies, as if they were distributed as follows:

Step	If the Range of the Distribution Covers 25 Steps, Per Cent.	If the Range of the Distribution Covers 20 Steps, Per Cent.	If the Range of the Distribution Covers 15 Steps, Per Cent.	If the Range of the Distribution Covers 12 Steps, Per Cent.	If the Range of the Distribution Covers 8 Steps, Per Cent.
0 to .1	9.5	9	8.5	9	8
.1 to .2	9.5	9.5	9	9	8
.2 to .3	9.5	9.5	9	9	9
.3 to .4	10	10	10	9	9
.4 to .5	10	10	10	10	10
.5 to .6	10	10	10	10	10
.6 to .7	10	10	10.5	10	11
.7 to .8	10.5	10.5	11	11	11
.8 to .9	10.5	10.5	11	11	12
.9 to 1.0	10.5	11	11	12	12

Consider the cases within the step where the 75 percentile lies as distributed as follows:

Step	Per Cent.
0 to .1	11
.1 to .2	11
.2 to .3	11
.3 to .4	10
.4 to .5	10
.5 to .6	10
.6 to .7	10
.7 to .8	9
.8 to .9	9
.9 to 1.0	9

Or, more exactly, in dependence on the coarseness of grouping, consider the cases, within the step where the 75 percentile lies, as distributed as follows:

Step	If the Range of the Distribution Covers 25 Steps, Per Cent	If the Range of the Distribution Covers 20 Steps, Per Cent.	If the Range of the Distribution Covers 15 Steps, Per Cent.	If the Range of the Distribution Covers 12 Steps, Per Cent.	If the Range of the Distribution Covers 8 Steps, Per Cent.
0 to .1	10.5	11	11	12	12
.1 to .2	10.5	10.5	11	11	12
.2 to .3	10.5	10.5	11	11	11
.3 to .4	10	10	10.5	10	11
.4 to .5	10	10	10	10	10
.5 to .6	10	10	10	10	10
.6 to .7	10	10	10	9	9
.7 to .8	9.5	9.5	9	9	9
.8 to .9	9.5	9.5	9	9	8
.9 to 1.0	9.5	9	8.5	9	8

§ 11. *Estimating the Central Tendency and Variability of the Entire Surface of Frequency, on the Basis of n Samples Taken at Random from its Total Numbers of Measures, N*

Besides getting what will probably be more accurate estimates of central tendency and of variability in view of what the measures within any one scale-interval would have been with a finer grouping, it is often desirable to estimate the central tendency and variability of the fact, supposing that many more cases of it had been taken. Thus suppose that a record includes 100 reaction-times of individual *A*. We are less interested in these 100 cases *per se*, than in them as a random sampling of the fact, *A*'s reaction time. It is desirable to estimate the probable central tendency, variability and form of distribution of the indefinite group of reaction times of which these are a limited sample.

Consider then the problem of estimating the central tendency, variability and form of distribution of the surface of frequency of all possible cases (N) of a fact of which n cases only are actually reported. In the case of the average, median, A.D., S.D., and percentile measures there need be no change from the procedures of Sections 9 and 10.

In the case of the mode, the issue changes from finding the measure of greatest frequency and choosing a point within it to best represent it. It is now to estimate that point on the scale at which the curve of frequency bounding the surface for the entire N cases is highest. This is too difficult a problem for exact solution by any save expert students of mathematics. A rough approximation toward such a solution may be got by drawing a smooth curve to fit the observed bounding-line of the surface of frequency and noting the location of its highest point.

In the case of the Med. Dev. or P.E., if the distribution is approximately of the form shown in Fig. 12 (page 32), the formula Med. Dev. (or P.E.) = .6745 S.D. may be used to give the probable median deviation from their central tendency of the entire *N* cases of which the *n* cases observed are a random sample.

§ 12. *Summary of Procedures for Ordinary Statistical Work*

It is more important for the worker with measurements to know just what he is doing in any case and why he does it than to follow any rigid conventions. But for the student who has mastered the reasons why, with a quantity varying continually or, if discrete, reported more coarsely than by its ultimate steps, it is profitable to take account of the probable spreading of the cases with finer grouping, the following recommendations as to general usages will be serviceable.

1. *For truly discrete measures, so reported,* use face-value methods.[10]

2. *For continuous or pseudo-continuous measures:*

For the Av.—Treat the *m*'s of each step as at its mid-point.

For the A.D.—Treat the *m*'s of each step as at its mid-point, correcting for coarse grouping by the table on page 55 if necessary.[11]

For the S.D.—Treat the *m*'s of each step as at its mid-point, correcting for coarse grouping by the table on page 55.[11]

For the Median—Treat the *m*'s of each step as spread evenly over it.

For the 25 percentile—Treat the *m*'s of each step as noted on page 57.

For the 75 percentile—Treat the *m*'s of each step as noted on page 58.

For the *Q*—Use the corrected values of the 25 and 75 percentile as obtained above, in the formula $Q = (75 \text{ percentile} - 25 \text{ percentile})/2$; or, in the same formula, use the values of the 25 percentile and 75 percentile obtained by taking the *m*'s of each step as spread evenly over it.[12]

---

[10] Beside the actual distribution and its crude mode, the most probable distribution for all cases of the fact (Dis.$_N$) and the most probable point-mode for Dis.$_N$ may be added. Such an addition is, however, inadvisable for such workers as are likely to use this book, because of the great difficulty of estimating the most probable Dis.$_N$ from the actual Dis.$_n$.

[11] Obviously the correction for coarse grouping need not be made, if it does not affect the significant figures in the result.

For the Median Deviation or P.E.—Use Q as an approximation, stating the fact, or, in distributions of approximately the form of Fig. 12, use .6745 S.D.

For the Mode—Use Mode = Av. − 3 (Av. − Median).

For the Skewness—Use Skewness $= \frac{3(\text{Av.} - \text{Median})}{\text{S.D.}}$

3. For continuous or pseudo-continuous measures it is entirely justifiable to *neglect all corrections for the form of distribution.* One may claim that facts outside a measure itself should not be allowed to influence our interpretation of it, and that consequently in continuous and pseudo-continuous measures, the measures within any one step should be treated as spread evenly over it. On this basis:

For the Av. A.D. and S.D.—Treat the *m*'s of each step as at its mid-point.

For the Median, all Percentiles, Q and the Median Deviation—Treat the *m*'s of each step as spread evenly over it.

Correct the S.D. by the formula:

$$\text{S.D.} = \sqrt{(\text{S.D.}_{\text{mid.}})^2 - \frac{1}{12}}$$

Problems

13. Calculate the 25 percentile, Median, 75 percentile, Q, Average, A.D. from Av., and σ (S.D.) from Av., for each of the following series of measures, assuming that the measures are discrete, and supposing that step 1 = 21; step 2 = 22; step 3 = 23, etc. (See the instructions following problem 22.)

---

[12] Since the error due to the form of the distribution acts in opposite directions upon the 25 percentile and the 75 percentile, this much quicker procedure is precise enough in the case of distributions which are approximately symmetrical.

Series I

		Frequency
Step	1	1
“	2	3
“	3	1
“	4	3
“	5	4
“	6	4
“	7	10
“	8	13
“	9	13
“	10	18
“	11	16
“	12	9
“	13	15
“	14	20
“	15	10
“	16	6
“	17	7
“	18	3
“	19	1
“	20	2
“	21	2
“	22	2
“	23	0
“	24	2

Series II

		Frequency
Step	1	2
“	2	0
“	3	2
“	4	2
“	5	6
“	6	3
“	7	10
“	8	12
“	9	17
“	10	28
“	11	16
“	12	30
“	13	25
“	14	30
“	15	22
“	16	23
“	17	23
“	18	13
“	19	11
“	20	11
“	21	11
“	22	2
“	23	1
“	24	4
“	25	5
“	26	0
“	27	1
“	28	0
“	29	1
“	30	0
“	31	0
“	32	0
“	33	1

Series III

		Frequency
Step	1	1
“	2	1
“	3	4
“	4	7
“	5	13
“	6	20
“	7	22
“	8	15
“	9	5
“	10	2
“	11	2

Series IV

		Frequency
Step	1	2
“	2	1
“	3	4
“	4	9
“	5	21
“	6	11
“	7	6
“	8	1
“	9	1

14. Change the values obtained above for Series I. to fit the supposition that steps 1, 2, 3, etc., of Series I. have the values 10, 12, 14, etc.

15. Change the values obtained above for Series I. to fit the supposition that the steps 1, 2, 3, etc., of the scale have the values 60, 66, 72, etc.,

16. Change the values obtained above for Series I. to fit the supposition that the steps 1, 2, 3, etc., of the scale have the values, − 8, − 6, − 4, − 2, 0, + 2, etc.

17. Calculate the 25 percentile, median, 75 percentile and Q for each of the four series, assuming (1) that the measures are continuous, (2) that the m's within any step are spread evenly over it, and (3) that the steps, 1, 2, 3, etc., have the values 20.5 to 21.5, 21.5 to 22.5, 22.5 to 23.5, etc.

18. Change the values obtained in problem 17 for Series I to fit these suppositions: (1) and (2) of 17, and (3) that the steps 1, 2, 3, etc., have the values 21.0 to 22.0, 22.0 to 23.0, 23.0 to 24.0, etc.

19. Change the values obtained in problem 17 for Series I. to fit these suppositions: (1) and (2) of 17, and (3) that the steps, 1, 2, 3, etc., have the values 30.0 to 35.0, 35.0 to 40.0, 40.0 to 45.0, etc.

20. Record any necessary changes in the values obtained in 13 for the Av., Av. A.D. from Av. and σ from Av. for each series, to fit the suppositions stated in 17.

21. Record any necessary changes in the values obtained in 13 for the Av., A.D. from Av. and σ from Av. to fit these suppositions: (1) and (2) of 17, and (3) of 18.

22. Same as 21, save that the suppositions concerning the steps of the scale are to be as follows:

In I., let steps 1, 2, 3, etc., be 17.9 to 18.1, 18.1 to 18.3, 18.3 to 18.5, etc.
" II., " " " " " 40.0 to 46.0, 46.0 to 52.0, 52.0 to 58.0, etc.
" III., " " " " " $8\frac{1}{3}$ to $8\frac{2}{3}$, $8\frac{2}{3}$ to 9, 9 to $9\frac{1}{3}$, etc.
" IV., " " " " " a to $a+k$, $a+k$ to $a+2k$, $a+2k$ to $a+3k$.

In problems 13 to 22, inclusive, be sure to work according to the short methods described in this chapter. Otherwise the computations will be very long. In choosing an approximate average, use the information gained in calculating the median. In recording results, follow a systematic arrangement, such as that reproduced in part on page 63. Make all answers to 13 and 17 accurate to the second decimal place. Accuracy to the first decimal place is all that is required for the others.

In the work of computation a table of products and a table of squares are aids to speed and precision. Crelle's *Rechentafeln* and *Barlow's Tables* are standard works. Shorter tables will be found in Appendix II of this volume.

SERIES I

	Assumed to be Discrete					Assumed to be Continuous			
	Scale being called 1, 2, 3, 4, etc.	Scale being called 21, 22, 23, 24, etc.	Scale being called 10, 12, 14, etc.	Scale being called 60, 66, 72, etc.	Scale being called —8, —6, —4, etc.	Scale being called 20.5 to 21.5, 21.5 to 22.5, etc.	Scale being called 21.0 to 22.0, 22.0 to 23.0, etc.	Scale being called 30 to 35, 35 to 40, etc.	Scale being called 17 9 to 18.1, 18.1 to 18.3, etc.
25 percentile									
Median									
75 percentile									
Q									
Approx. Av. (G.A.)									
d									
Av.									
$\Sigma\xi$									
Σx									
A.D. from Av.									
$\Sigma\xi^2$									
σ from Av.									

CHAPTER V

TECHNICAL AIDS IN DESCRIBING AND CONSTRUCTING THE FORM OF A SURFACE OF FREQUENCY

§ 13. *Graphs and Equations of the Bounding Line*

THE form of a surface of frequency is defined most easily by presenting it graphically, as has so far been done in this volume and as is done in Figs. 15 to 23 in this chapter. Such a graphic description is often the only measurement of the geometrical form of a surface of frequency that is practicable, especially for the non-mathematical student.

The form of the surface of frequency is also definable by a geometrical or algebraical description of the line, which, together with the base-line, bounds it. Thus, if x = any given point of the horizontal scale along which the trait in question is measured and y = the height of the surface of distribution at that point, the equation of the line bounding a rectangular surface of frequency is:

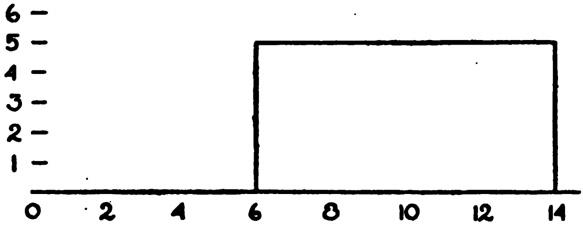

FIG. 15.

$$y = K \text{ for values of } x \text{ from } A \text{ to } B,$$
$$y = 0 \text{ for all other values of } x,$$

in which K is a constant, the altitude of the rectangle,

A is the distance from 0 of the x scale to one extreme of the base of the rectangle, and

B is the distance from 0 of the x scale to the other extreme of the base of the rectangle.

Thus, in Fig. 15, K being 5, A being 6, and B being 14, the equation of the bounding line of the surface of frequency is:

$y = 5$ for values of x from 6 to 14,
$y = 0$ for all other values of x.

Similarly the equation of the line bounding the surface of frequency of a right triangle with base coinciding with the x scale, and with its highest point nearest to the zero-point of the x scale, is:

$y = K\ (B - x)$ within the limits of $x = A$ and $x = B$,
$y = 0$ for all other values of x,

in which K is a constant—the altitude of the triangle, divided by its base,
A is the distance from 0 of the x scale to the vertex of the right angle, and
B is the distance from 0 of the x scale to the other end of the base of the triangle.

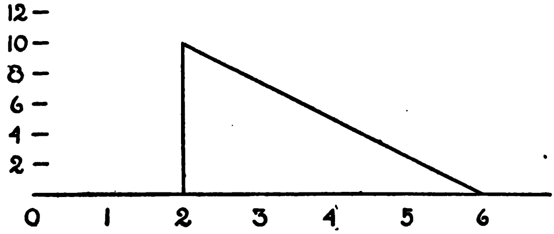

Fig. 16.

Thus, in Fig. 16, the altitude being 10, A being 2 and B being 6, and K consequently 2.5, the equation of the bounding line of the surface of frequency is:

$y = 2.5\ (6 - x)$ for values of x from 2 to 6, and
$y = 0$ for all other values of x.

Similarly the equation of the line bounding the surface of frequency of an isosceles triangle with base coincident with the x scale is:

$y = K\ (x - A)$ for values of x from A to C,
$y = K\ (B - x)$ for values of x from C to B,
$y = 0$ for all other values of x.

in which K is a constant—the altitude divided by one half of the base,

A is the distance from 0 of the x scale to the end of the base nearest the zero-point of the x scale,
B is the distance from 0 of the x scale to the other end of the base, and
C is the distance from 0 of the x scale to the middle of the base.

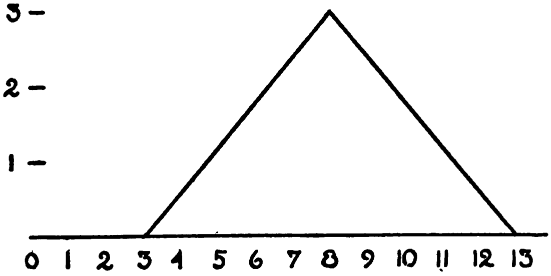

FIG. 17.

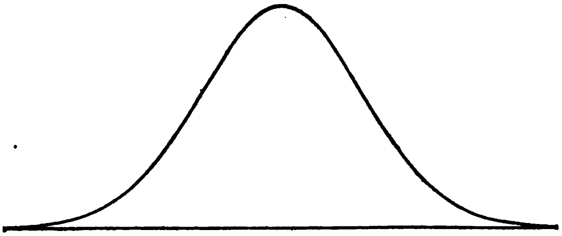

FIG. 18. Surface of Frequency of Form A.

Thus, in Fig. 17, the altitude being 3, *A* being 3, *B* being 13, *C* being 8, and *K* consequently being 2, the equation is:

$y = 3/5\ (x - 3)$ for values of x from 3 to 8,
$y = 3/5\ (13 - x)$ for values of x from 8 to 13,
$y = 0$ for all other values of x.

The equation of the line bounding the surface of frequency of the form shown in Fig. 18 (see also Fig. 12 on page 32) is, the zero-point of the x scale being taken in this case, not at its absolute zero, but at the point where y is greatest:

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

This curve is called the curve of error or the probability curve, and the surface of frequency which it encloses (with the base line) is called the "normal" distribution or "normal" surface of frequency or the surface of frequency of the normal probability integral. This last case is one of special importance to the theory of measurements for several reasons. One of these, which has already been noted, is that the distributions actually found for variable facts often approximate more closely to it than to a rectangle, an isosceles triangle, or any other simple geometrical form. Other reasons will appear later in this volume. I shall refer to this form of distribution—that defined by the equation

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

as *Form A*.[1]

In this form of distribution the Average, Median and Mode coincide, for y is the same for any given $-x$ as for the same $+x$, and is greatest when $x = 0$. Constant relations hold between the different measures of variability, *viz:*

$$\begin{aligned} \sigma &= 1.25331 \text{ A.D.} \\ \sigma &= 1.4825 \text{ P.E.} \\ \text{A.D.} &= .7979\,\sigma \\ \text{A.D.} &= 1.1843 \text{ P.E.} \\ \text{P.E.} &= .6745\,\sigma \\ \text{P.E.} &= .8453 \text{ A.D.} \end{aligned}$$

Between Av. $-\ \sigma$ and Av. $+\ \sigma$ are 68.26 per cent. of the cases.
" Av. − A.D. and Av. + A.D. are 57.5 per cent. of the cases.
" Av. − P.E. " Av. + P.E. " 50 " " " " "

Fig. 19 shows a distribution much skewed, which we may call Form C. Fig. 20 shows a distribution still more skewed, which we may call Form D. The bounding lines of Fig. 19 and Fig. 20 can not be represented by any simple equations.

§ 14. *Tables of Frequency*

The form of a surface of frequency is definable also by tables which give, directly or indirectly, the relative frequencies of different amounts of the trait. Thus Tables 7 and 8 tell in different ways the fact that the form of distribution is a rectangle. They do not, it should be observed, tell whether the average is at 1, — 10, — 60 or 0.492, or anything else about the average, save, of course, that it is at the mid-point of the base. They do not tell anything about the amount of variability or dispersion. The rectangle may stretch from 99 to 101 or from 60 to 6,000 or from .92 to .925. They tell only its geometrical form.

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[1] The student need not concern himself with this equation further than to accept the fact that it is the equation of the bounding line of Figure 18. No use of the equation itself will be required.

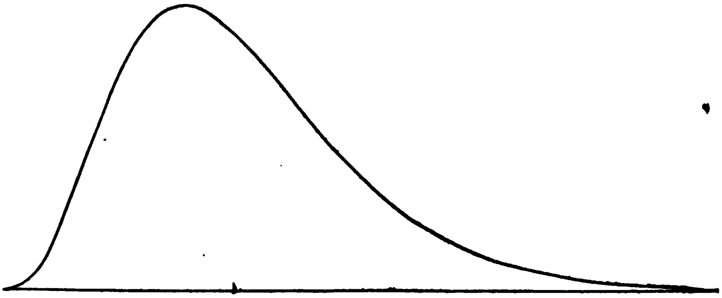

Fig. 19. Surface of Frequency of Form C.

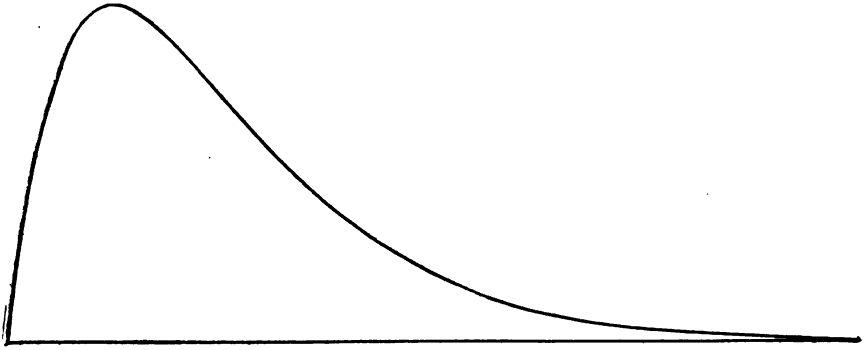

Fig. 20. Surface of Frequency of Form D.

These tables for showing the form of distribution of a rectangle are of no utility, since anyone could quickly construct any one of them; but the tables which follow (Tables 9, 10 and 11) are useful. They tell approximately in figures the facts represented graphically in Figs. 18, 19 and 20, and so give tabular representations of the normal probability surface, a surface much skewed, and a surface very much skewed, all three being forms to some one of which an actual distribution is likely to approximate. They repeat exactly Figs. 21, 22 and 23, which duplicate, in approximations by rectangles, Figs. 18, 19 and 20, in order. In each of these last three diagrams, the total area of the surface of frequency is 10 square inches[2] and each small division of the base line is .1 σ (one tenth of the mean square deviation of the distribution in question).

TABLE 7

Relative Frequencies (in Per Cents.) over Each Tenth of Q

Quantity	Frequency
$-2.2\,Q$ to $-2.1\,Q$	0
$-2.1\,Q$ to $-2.0\,Q$	0
$-2.0\,Q$ to $-1.9\,Q$	2.5
$-1.9\,Q$ to $-1.8\,Q$	2.5
$-1.8\,Q$ to $-1.7\,Q$	2.5
$-1.7\,Q$ to $-1.6\,Q$	2.5
$-1.6\,Q$ to $-1.5\,Q$	2.5
$-1.5\,Q$ to $-1.4\,Q$	2.5
$-1.4\,Q$ to $-1.3\,Q$	2.5
$-1.3\,Q$ to $-1.2\,Q$	2.5
$-1.2\,Q$ to $-1.1\,Q$	2.5
$-1.1\,Q$ to $-1.0\,Q$	2.5
$-1.0\,Q$ to $-.9\,Q$	2.5
$-.9\,Q$ to $-.8\,Q$	2.5
$-.8\,Q$ to $-.7\,Q$	2.5
$-7.\,Q$ to $-.6\,Q$	2.5
$-.6\,Q$ to $-.5\,Q$	2.5
$-.5\,Q$ to $-.4\,Q$	2.5
$-.4\,Q$ to $-.3\,Q$	2.5
$-.3\,Q$ to $-.2\,Q$	2.5
$-.2\,Q$ to $-.1\,Q$	2.5
$-.1\,Q$ to 0	2.5
$0\,Q$ to $+.1\,Q$	2.5
$+.1\,Q$ to $+.2\,Q$	2.5
$+.2\,Q$ to $+.3\,Q$	2.5
$+.3\,Q$ to $+.4\,Q$	2.5
$+.4\,Q$ to $+.5\,Q$	2.5
$+.5\,Q$ to $+.6\,Q$	2.5
$+.6\,Q$ to $+.7\,Q$	2.5
$+.7\,Q$ to $+.8\,Q$	2.5
$+.8\,Q$ to $+.9\,Q$	2.5
$+.9\,Q$ to $+1.0\,Q$	2.5
$+1.0\,Q$ to $+1.1\,Q$	2.5
$+1.1\,Q$ to $+1.2\,Q$	2.5
$+1.2\,Q$ to $+1.3\,Q$	2.5
$+1.3\,Q$ to $+1.4\,Q$	2.5
$+1.4\,Q$ to $+1.5\,Q$	2.5
$+1.5\,Q$ to $+1.6\,Q$	2.5
$+1.6\,Q$ to $+1.7\,Q$	2.5
$+1.7\,Q$ to $+1.8\,Q$	2.5
$+1.8\,Q$ to $+1.9\,Q$	2.5
$+1.9\,Q$ to $+2.0\,Q$	2.5
$+2.0\,Q$ to $+2.1\,Q$	0
$+2.1\,Q$ to $+2.2\,Q$	0

TABLE 8

Relative Frequencies (in Per Cents.) over Each Tenth of σ

Quantity	Frequency
$-1.9\,\sigma$ to $-1.8\,\sigma$	.000
$-1.8\,\sigma$ to $-1.7\,\sigma$	.938
$-1.7\,\sigma$ to $-1.6\,\sigma$	2.886
$-1.6\,\sigma$ to $-1.5\,\sigma$	2.886
$-1.5\,\sigma$ to $-1.4\,\sigma$	2.886
$-1.4\,\sigma$ to $-1.3\,\sigma$	2.886
$-1.3\,\sigma$ to $-1.2\,\sigma$	2.886
$-1.2\,\sigma$ to $-1.1\,\sigma$	2.886
$-1.1\,\sigma$ to $-1.0\,\sigma$	2.886
$-1.0\,\sigma$ to $-.9\,\sigma$	2.886
$-.9\,\sigma$ to $-.8\,\sigma$	2.886
$-.8\,\sigma$ to $-.7\,\sigma$	2.886
$-.7\,\sigma$ to $-.6\,\sigma$	2.886
$-.6\,\sigma$ to $-.5\,\sigma$	2.886
$-.5\,\sigma$ to $-.4\,\sigma$	2.886
$-.4\,\sigma$ to $-.3\,\sigma$	2.886
$-.3\,\sigma$ to $-.2\,\sigma$	2.886
$-.2\,\sigma$ to $-.1\,\sigma$	2.886
$-.1\,\sigma$ to 0	2.886
$0\,\sigma$ to $+.1\,\sigma$	2.886
$+.1\,\sigma$ to $+.2\,\sigma$	2.886
$+.2\,\sigma$ to $+.3\,\sigma$	2.886
$+.3\,\sigma$ to $+.4\,\sigma$	2.886
$+.4\,\sigma$ to $+.5\,\sigma$	2.886
$+.5\,\sigma$ to $+.6\,\sigma$	2.886
$+.6\,\sigma$ to $+.7\,\sigma$	2.886
$+.7\,\sigma$ to $+.8\,\sigma$	2.886
$+.8\,\sigma$ to $+.9\,\sigma$	2.886
$+.9\,\sigma$ to $+1.0\,\sigma$	2.886
$+1.0\,\sigma$ to $+1.1\,\sigma$	2.886
$+1.1\,\sigma$ to $+1.2\,\sigma$	2.886
$+1.2\,\sigma$ to $+1.3\,\sigma$	2.886
$+1.3\,\sigma$ to $+1.4\,\sigma$	2.886
$+1.4\,\sigma$ to $+1.5\,\sigma$	2.886
$+1.5\,\sigma$ to $+1.6\,\sigma$	2.886
$+1.6\,\sigma$ to $+1.7\,\sigma$	2.886
$+1.7\,\sigma$ to $+1.8\,\sigma$	.938
$+1.8\,\sigma$ to $+1.9\,\sigma$	0
$+1.9\,\sigma$ to $+2.0\,\sigma$	0

---

[2] Except that, in Fig. 21, .013 sq. in. at each extreme is not shown in the diagram.

	TABLE 9	TABLE 10	TABLE 11
	Relative Frequencies (in Percentages) Over Each Tenth of σ, in a Surface of Frequency of Form *A*	**Relative Frequencies (in Percentages) Over Each Tenth of σ, in a Surface of Frequency of Form *C***	**Relative Frequencies (in Percentages) Over Each Tenth of σ, in a Surface of Frequency of Form *D***
−4.2 σ to −4.1 σ	.001		
−4.1 σ to −4.0 σ	.001		
−4.0 σ to −3.9 σ	.002		
−3.9 σ to −3.8 σ	.002		
−3.8 σ to −3.7 σ	.004		
−3.7 σ to −3.6 σ	.005		
−3.6 σ to −3.5 σ	.007		
−3.5 σ to −3.4 σ	.010		
−3.4 σ to −3.3 σ	.015		
−3.3 σ to −3.2 σ	.02		
−3.2 σ to −3.1 σ	.03		
−3.1 σ to −3.0 σ	.04		
−3.0 σ to −2.9 σ	.05		
−2.9 σ to −2.8 σ	.07		
−2.8 σ to −2.7 σ	.09		
−2.7 σ to −2.6 σ	.12		
−2.6 σ to −2.5 σ	.15		
−2.5 σ to −2.4 σ	.20		
−2.4 σ to −2.3 σ	.25		
−2.3 σ to −2.2 σ	.32		
−2.2 σ to −2.1 σ	.40		
−2.1 σ to −2.0 σ	.49		
−2.0 σ to −1.9 σ	.60		
−1.9 σ to −1.8 σ	.72		
−1.8 σ to −1.7 σ	.86		
−1.7 σ to −1.6 σ	1.02		
−1.6 σ to −1.5 σ	1.20		
−1.5 σ to −1.4 σ	1.39		
−1.4 σ to −1.3 σ	1.60	.01	
−1.3 σ to −1.2 σ	1.83	.10	
−1.2 σ to −1.1 σ	2.06	.28	
−1.1 σ to −1.0 σ	2.30	.60	
−1.0 σ to − .9 σ	2.54	1.14	
− .9 σ to − .8 σ	2.78	1.76	
− .8 σ to − .7 σ	3.01	2.38	
− .7 σ to − .6 σ	3.23	2.94	.03
− .6 σ to − .5 σ	3.43	3.49	1.38
− .5 σ to − .4 σ	3.60	3.99	3.34
− .4 σ to − .3 σ	3.75	4.36	4.50
− .3 σ to − .2 σ	3.87	4.63	5.34
− .2 σ to − .1 σ	3.94	4.80	5.85
− .1 σ to 0 σ	3.98	4.89	6.05

	TABLE 9 *(continued)* Relative Frequencies (in Percentages) Over Each Tenth of σ, in a Surface of Frequency of Form *A*	TABLE 10 *(continued)* Relative Frequencies (in Percentages) Over Each Tenth of σ, in a Surface of Frequency of Form *C*	TABLE 11 *(continued)* Relative Frequencies (in Percentages) Over Each Tenth of σ, in a Surface of Frequency of Form *D*
0σ to $+.1\sigma$	3.98	4.89	6.05
$+.1\sigma$ to $+.2\sigma$	3.94	4.80	5.92
$+.2\sigma$ to $+.3\sigma$	3.87	4.65	5.67
$+.3\sigma$ to $+.4\sigma$	3.75	4.45	5.36
$+.4\sigma$ to $+.5\sigma$	3.60	4.26	5.02
$+.5\sigma$ to $+.6\sigma$	3.43	4.06	4.65
$+.6\sigma$ to $+.7\sigma$	3.23	3.80	4.30
$+.7\sigma$ to $+.8\sigma$	3.01	3.52	3.94
$+.8\sigma$ to $+.9\sigma$	2.78	3.25	3.62
$+.9\sigma$ to $+1.0\sigma$	2.54	2.99	3.31
$+1.0\sigma$ to $+1.1\sigma$	2.30	2.73	3.00
$+1.1\sigma$ to $+1.2\sigma$	2.06	2.48	2.69
$+1.2\sigma$ to $+1.3\sigma$	1.83	2.24	2.41
$+1.3\sigma$ to $+1.4\sigma$	1.60	2.02	2.19
$+1.4\sigma$ to $+1.5\sigma$	1.39	1.81	1.95
$+1.5\sigma$ to $+1.6\sigma$	1.20	1.62	1.73
$+1.6\sigma$ to $+1.7\sigma$	1.02	1.43	1.55
$+1.7\sigma$ to $+1.8\sigma$	.86	1.26	1.37
$+1.8\sigma$ to $+1.9\sigma$	.72	1.12	1.19
$+1.9\sigma$ to $+2.0\sigma$	.60	.98	1.05
$+2.0\sigma$ to $+2.1\sigma$	.49	.87	.93
$+2.1\sigma$ to $+2.2\sigma$	.40	.77	.80
$+2.2\sigma$ to $+2.3\sigma$	.32	.67	.69
$+2.3\sigma$ to $+2.4\sigma$	.25	.58	.60
$+2.4\sigma$ to $+2.5\sigma$	.20	.50	.52
$+2.5\sigma$ to $+2.6\sigma$	.15	.44	.46
$+2.6\sigma$ to $+2.7\sigma$	.12	.39	.39
$+2.7\sigma$ to $+2.8\sigma$	.09	.34	.33
$+2.8\sigma$ to $+2.9\sigma$	.07	.29	.27
$+2.9\sigma$ to $+3.0\sigma$	.05	.25	.23
$+3.0\sigma$ to $+3.1\sigma$	.04	.21	.21
$+3.1\sigma$ to $+3.2\sigma$	.03	.18	.19
$+3.2\sigma$ to $+3.3\sigma$	.02	.16	.17
$+3.3\sigma$ to $+3.4\sigma$	.015	.14	.15
$+3.4\sigma$ to $+3.5\sigma$	.010	.12	.14
$+3.5\sigma$ to $+3.6\sigma$	.007	.10	.12
$+3.6\sigma$ to $+3.7\sigma$	.005	.08	.10
$+3.7\sigma$ to $+3.8\sigma$	.004	.07	.08
$+3.8\sigma$ to $+3.9\sigma$	.002	.05	.07
$+3.9\sigma$ to $+4.0\sigma$	.002	.03	.05
$+4.0\sigma$ to $+4.1\sigma$	.001	.015	.03
$+4.1\sigma$ to $+4.2\sigma$	.001	.005	.015
$+4.2\sigma$ to $+4.3\sigma$			.005

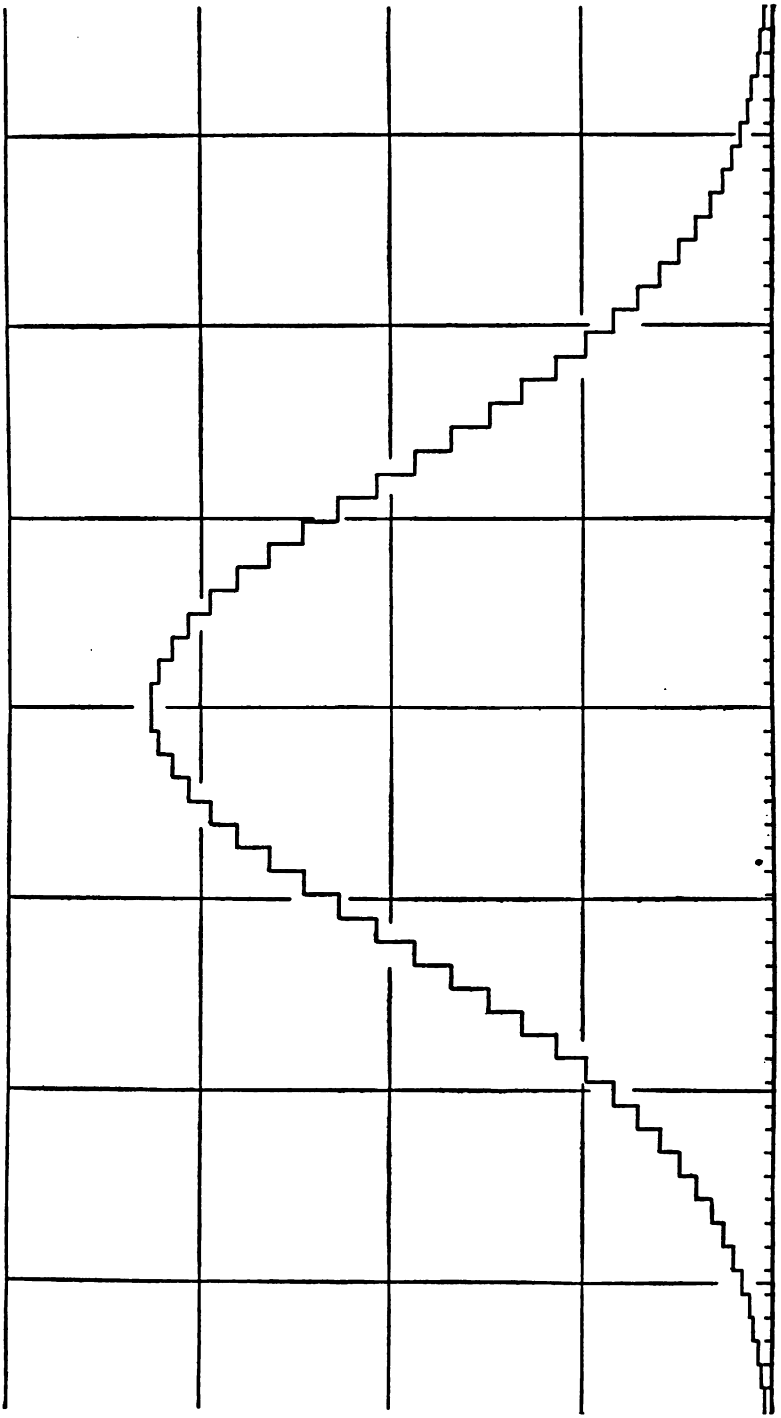

FIG. 21. Surface of Frequency of Form A.

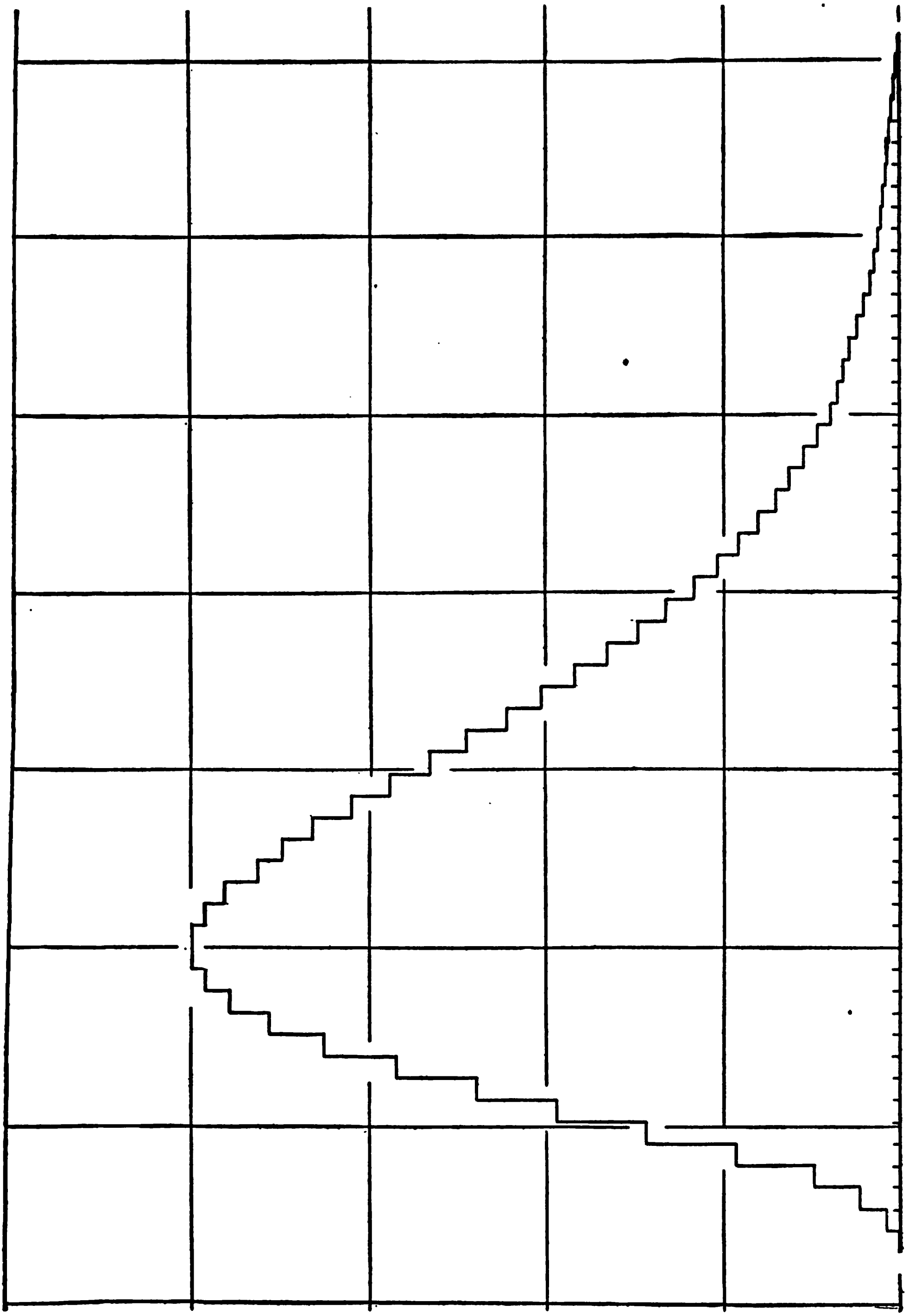

FIG. 22. Surface of Frequency of Form C.

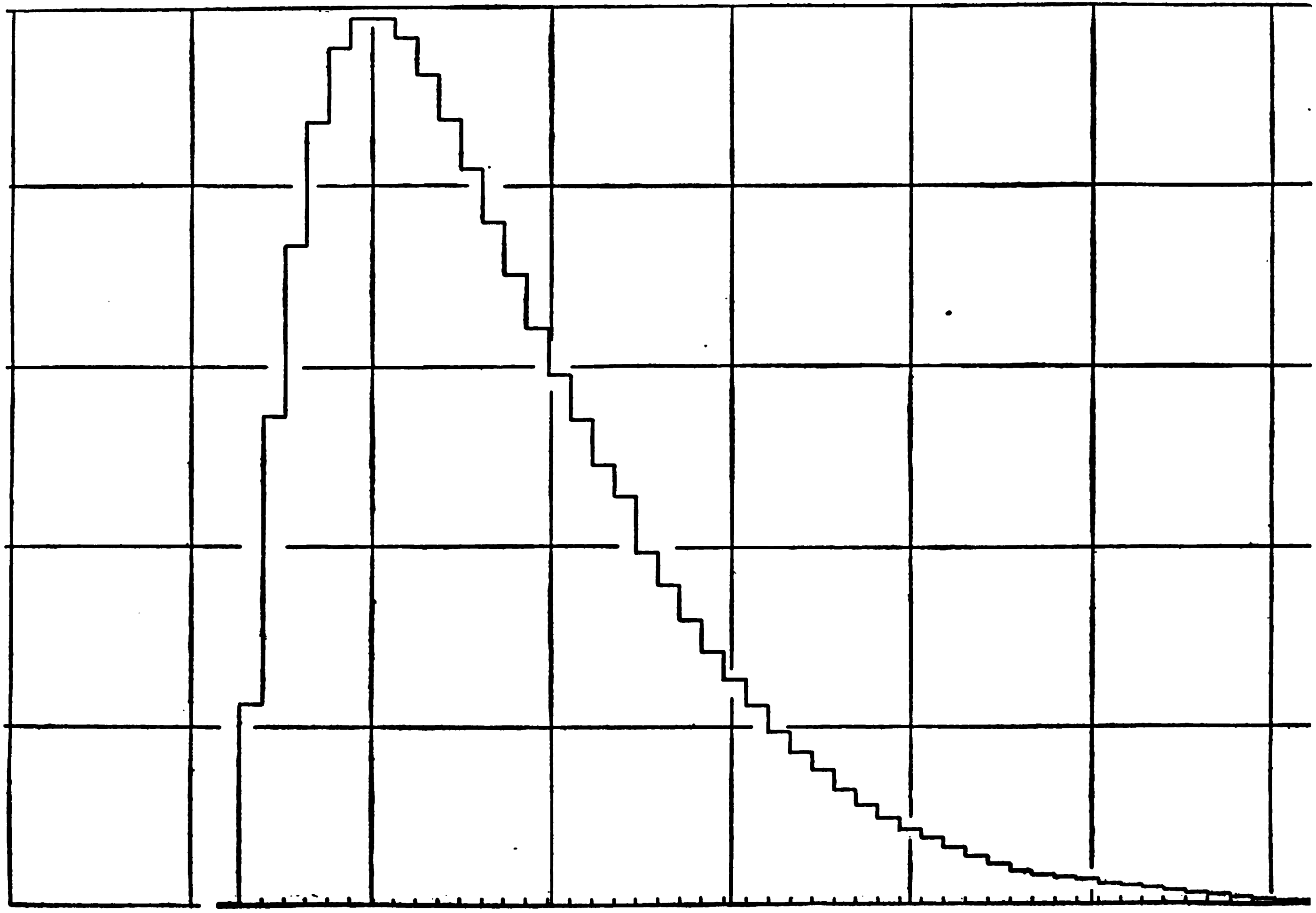

FIG. 23. Surface of Frequency of Form D.

The form of certain obtained distributions may then be defined roughly by means of graphic comparison with Figs. 18–20 or 21–23, or by means of a comparison of the table of frequencies in question with Tables 9, 10 and 11. Graphic comparison by drawing the surface in question, so scaled that its total area equals 10 sq. inches and the σ approx. 1.225 inch, over one of the surfaces shown in Figs. 21, 22 and 23 is perhaps the more convenient for ordinary practise. The mode of the surface in question should be made to coincide approximately with the mode of Fig. 21, 22, or 23 as the case may be. (In the case of comparison with Fig. 21 the median will serve better than the mode.) Tabular comparison requires that the distribution in question be put in percentages (to a first decimal), and that a table be constructed from Table 9, 10 or 11, as the case may be, with approximately the same fraction of the variability as the step as is the case in the distribution in question. This may involve a tedious, though straightforward, set of computations.

As an illustration of graphic comparison we may take the following: Required to describe the form of distribution of the fact shown in Table 12.

TABLE 12

RATIO OF ATTENDANCE TO ENROLLMENT REPORTED IN AMERICAN CITIES IN 1902

Quantity	Gross Frequency	Frequency in Percentages
43 to 47	1	.18
47 to 51	1	.18
51 to 55	2	.37
55 to 59	5	.92
59 to 63	6	1.10
63 to 67	36	6.61
67 to 71	78	14.3
71 to 75	85	15.6
75 to 79	154	28.3
79 to 83	107	19.6
83 to 87	44	8.07
87 to 91	20	3.67
91 to 95	3	.55
95 to 99	3	.55
	$n = 545$	

The distribution of Table 12, being drawn to the same area as Fig. 21 and with its S.D. approximately equal on the base line to the S.D. of Fig. 21, we have the dotted line of Fig. 24. When it is similarly fitted to Fig. 22 (first being reversed to get the better fit) we have the dotted line of Fig. 25. For convenience, the gross dimensions are reduced in both cases. The form of distribution of Table 12 is obviously not a close fit to either Form *A* or Form *C*.

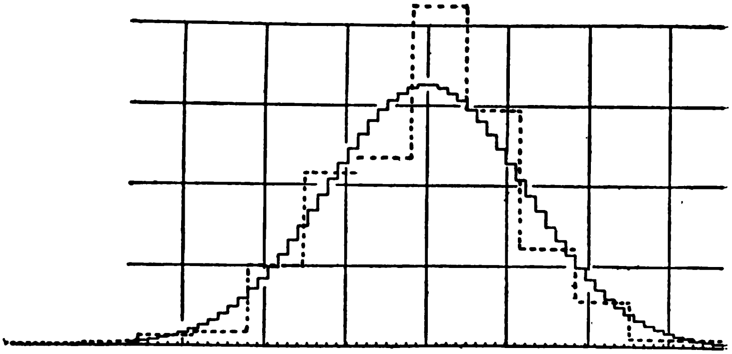

Fig. 24. The form of distribution of Table 12 compared with Form A.

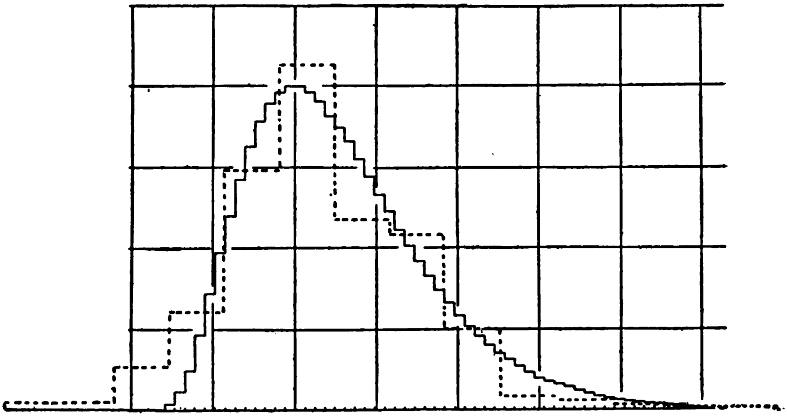

Fig. 25. The form of distribution of Table 12 compared with Form C.

Nor would it be to a form intermediate between the two. It is so irregular as to be best left to stand as its own description.

§ 15. *The Reconstruction of a Surface of Frequency from Knowledge of its Central Tendency, Variability and Form*

In certain cases two or three figures with a statement of the geometrical form of the surface of frequency enable one to reconstruct the entire surface of frequency or distribution table. Thus, "Av. = 10; $Q = 2$; form of distribution, a rectangle," tells us that the distribution is that enclosed by the continuous line of Fig. 26. So also "Av. = 10; $Q = 2$; *form of distribution, that of the surface of frequency of the normal probability integral,*" tells the student who is acquainted with certain facts that the distribution is that enclosed by the dotted line of Fig. 26.

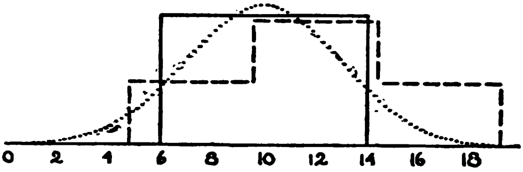

FIG. 26.

Similarly Av. 12; Median deviation 3.4; form of distribution a square of Z base, with two squares of $Z/2$ base adjoining it on each side, tells us that the distribution is that enclosed by the dash line of Fig. 26.

A change in the central tendency, variability and form being kept constant, pushes the whole distribution forward or back along the scale; a change in the variability, central tendency and form being kept constant, shrinks it in or expands it; a change in the form of the distribution, central tendency and variability being kept constant, makes certain measures more frequent and others less frequent without changing the point on the scale about which they cluster or their general amount of dispersion.

§ 16. *Skewness and Multimodality*

Skewness.—For one partial feature of the form of a surface of frequency, its skewness, conventional measures have been proposed. These are:

$$\text{Skewness} = \frac{(25 \text{ percentile}) + (75 \text{ percentile}) - 2\,(\text{Median})}{Q}$$

$$\text{Skewness} = \frac{2\,(\text{A.D.}_{+m} - \text{A.D.}_{-m})}{\text{A.D.}_{+m} + \text{A.D.}_{-m}}$$

in which A.D._{+m} = the average deviation from the median of the measures above the median and A.D._{-m} = the average deviation from the median of the measures below the median.

$$\text{Skewness} = \frac{3\,(\text{Av.} - \text{Median})}{\text{S.D.}}$$

These measures are all arbitrary, and any measure of the variability or dispersion of the measures night be used in any of the denominators. Some such convention has to be adopted if one is to compare different surfaces of frequency with respect to skewness numerically. The last is the approved one.

Multimodality.—Multimodal distributions may be merely graphed—or may be analyzed into the separate unimodal distributions out of which the investigator has reason to think they are compounded. No fixed rules for such analysis can be given here.

PROBLEMS

In Problem 23, use paper ruled horizontally and vertically with lines one tenth of an inch apart. Let one tenth of an inch linear equal always 1 unit of the scale for the quantity; and let the area of one square (.01 sq. in.) equal always one tenth of 1 per cent. of the frequencies.

23. Using Table 13, construct in the shape of a series of rectangles each on a base of .25σ approximations to 'normal' surfaces of frequency (that is, surfaces of Form A) to fit each of the following:

I. Central tendency = 40; σ = 8
II. Central tendency = 40; σ = 16
III. Central tendency = 40; σ = 4

TABLE 13

ONE OF THE SYMMETRICAL HALVES OF THE SURFACE OF FREQUENCY OF FORM A, GIVING THE PERCENTAGE OF THE TOTAL AREA FOR EACH INTERVAL OF .25 σ, STARTING AT THE CENTRAL TENDENCY

Quantity	Frequency
0 to .25 σ	9.87
.25 σ to .50 σ	9.28
.50 σ to .75 σ	8.19
.75 σ to 1.00 σ	6.80
1.00 σ to 1.25 σ	5.30
1.25 σ to 1.50 σ	3.88
1.50 σ to 1.75 σ	2.68
1.75 σ to 2.00 σ	1.73
2.00 σ to 2.25 σ	1.05
2.25 σ to 2.50 σ	.60
2.50 σ to 2.75 σ	.32
2.75 σ to 3.00 σ	.17
3.00 σ to 3.25 σ	.07
3.25 σ to 3.50 σ	.04
3.50 σ to 3.75 σ	.02

24. Draw, all on the same scale as base, the following surfaces of frequency, using different colors or kinds of lines to distinguish them. Make all surfaces have the same area by letting 10 square inches equal always 100 per cent. of the measures.

I. Av. = 6; $Q = 2$; form of distribution; a rectangle.
II. Av. = 6; $Q = 3$; form of distribution; a rectangle.
III. Av. = 8; $Q = 2.5$; form of distribution; a rectangle.

25(*a*). Assign amounts to 100 measures so that their surface of frequency will show about + .5 skewness—by the formula:

$$\text{Skewness} = \frac{3\,(\text{Av.} - \text{Median})}{\text{S.D.}}$$

25(*b*). Assign amounts to 100 measures so that their surface of frequency will show about − .25 skewness.

25(*c* and *d*). Arrange similarly a distribution of about + .75 skewness and one of about + 1.00 skewness.

CHAPTER VI

THE CAUSES OF VARIABILITY

THIS chapter aims to introduce the reader to an understanding of the nature of the causes which make a trait vary, which determine the extent and relative frequency of its variations, and which consequently determine the form of its distribution.

It has been shown in Chapter III. that the measures of a variable fact are often distributed approximately after the fashion of the surface of frequency enclosed by the probability curve and its abscissa. Brief mention has also been made of the properties of this form of distribution, acquaintance with which is a great assistance to convenient handling of mental measurements. The recognition of this type of frequency surface, the appreciation of its meaning and that of certain common departures from it, and the use of tables derived from it are all possible, at least to the moderate degree required for ordinary statistical work, without any knowledge of the abstract principles involved. But such knowledge is well worth obtaining for the sake of the additional insight into the meaning of concrete facts thereby given, and even merely for the sake of the additional facility in the use and construction of tables and the common formulæ. The present chapter will therefore consider especially the causes of variability in the case of distributions of Form *A*.

§ 17. *The Effect of Chance Combinations from Equally Potent Causes*

Let us begin with the consideration of a quantity which is dependent on the action of one cause which is as likely to occur as not, and call the cause *a*. For example, *a* may be the action of John's father in giving him a Christmas gift of a dollar.

The condition of affairs resulting will be, of course, no action or *a*. The quantity in question, John's Christmas money, will be 0 or $1.00. Its distribution will be

Quantity: Dollars	Frequency: Per Cent.
0	50
1	50

Its surface of frequency will be a rectangle, composed of two rectangles of equal base and altitude.

Suppose now that two causes contribute to determine the quantity, *a* and *b*, the possible actions of John's father and mother, and that all combinations of these causes are equally likely. The condition of affairs resulting will be, then, *no action*, *a*, *b* or *ab*, all being equally likely. If now $a =$ a gift of \$1.00 and *b* likewise, the quantity in question, John's Christmas money, will be 0, \$1.00, \$1.00 or \$2.00. Its distribution will be

Quantity: Dollars	Frequency: Per Cent.
0	25
1	50
2	25

Its surface of frequency is that shown in Fig. 27. If the conditions are kept the same but the number of causes increased to three, the condition of affairs will be, no action, *a*, *b*, *c*, *ab*, *ac*, *bc*, or *abc*. If as before $a = b = c$ in magnitude, John will get \$2.00 as often as \$1.00 and three times as often as nothing or \$3.00.

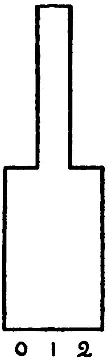

FIG. 27.

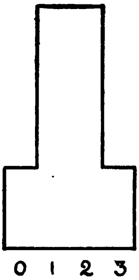

FIG. 28.

The surface of frequency of the quantity, John's Christmas money, will be four rectangles, as shown in Fig. 28.

Keeping all the conditions the same, let the number of causes be increased to 4, then to 5, and then to 6. The condition of affairs in each case and the resulting distribution-schemes and surfaces of frequency are given in Tables 14, 15 and 16, and Figs. 29, 30 and 31.

In these tables *ab, bde, abcd,* and the like mean $(a + b)$, $(b + d + e)$, $(a + b + c + d)$, etc., *not* $(a \times b)$, $(b \times d \times e)$, etc.

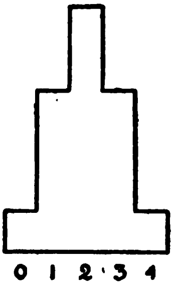

FIG. 29.

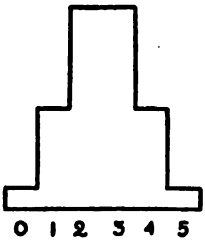

FIG. 30.

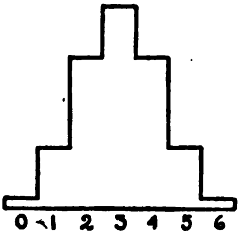

FIG. 31.

TABLE 14

COMBINATIONS OF 4 CAUSES, *a, b, c* AND *d*

	Value in Dollars	Probable Frequency
0	0	1
a, b, c, d	1.00	4
ab, ac, ad, bc, bd, cd	2.00	6
abc, abd, acd, bcd	3.00	4
abcd	4.00	1

TABLE 15

COMBINATIONS OF 5 CAUSES, *a, b, c, d* AND *e*

	Value in Dollars	Probable Frequency
0	0	1
a, b, c, d, e	1.00	5
ab, ac, ad, ae, bc, bd, be, cd, ce, de	2.00	10
abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde	3.00	10
abcd abce, abde, acde, bcde	4.00	5
abcde	5.00	1

TABLE 16

COMBINATIONS OF 6 CAUSES, *a, b, c, d, e* AND *f*

	Value in Dollars	Probable Frequency
0	0	1
a, b, c, d, e, f	1.00	6
ab, ac, ad, ae, af, bc, bd, be *bf, cd, ce, cf, de, df, ef*	2.00	15
abc, abd, abe, abf, acd, ace, acf *ade, adf, aef, bcd, bce, bcf, bde* *bdf, bef, cde, cdf, cef, def*	3.00	20
abcd, abce, abcf, abde, abdf *abef, acde, acdf, acef, adef* *bcde, bcdf, bcef, bdef, cdef*	4.00	15
abcde, abcdf, abcef, abdef, acdef, bcdef	5.00	6
abcdef	6.00	1

It is apparent that the surface of frequency of a quantity dependent upon the action of causes equal in magnitude, any combination of which is equally probable, tends, as the number of these causes becomes great, to approach Form *A*, the probability type. This is emphasized by Table 17 and Figs. 32, 33 and 34, which give the results in our illustration if the number of causes is increased to 10, 15 and 20 respectively. When the number of causes is infinite the result is exactly Form *A*.

TABLE 17

COMBINATIONS OF 10, 15 AND 20 CAUSES

Quantity: Dollars	Frequency in Case Of 10	Of 15	Of 20
0	1	1	1
1	10	15	20
2	45	105	190
3	120	455	1,140
4	210	1,365	4,845
5	252	3,003	15,504
6	210	5,005	38,760
7	120	6,435	77,520
8	45	6,435	125,970
9	10	5,005	167,960
10	1	3,003	184,756
11		1,365	167,960
12		455	125,970
13		105	77,520
14		15	38,760
15		1	15,504
16			4,845
17			1,140
18			190
19			20
20			1

The probability type of distribution may therefore be expected in the case of the different performances or measures of an individual in the same trait, if any one of his performances in the trait is due to the action of some one combination from a large number of causes of equal magnitude which are independent of each other, so that any combination is as likely to occur as any other; may be expected in the case of the different measures of individuals in a group, if the tendency of any individual in the trait is due to the action of some one combination, characteristic of his make-up, from such a large number of causes. If, that is, we think of any single act of a person as a result of a chance combination from amongst a number of causes which determine acts of that sort characteristic of him, we shall expect his different manifestations of the trait of which that act is a sample to follow Form *A*; so also, if we think of the quantity of a trait in any single individual of a group as a result of a chance combination from amongst a number of causes characteristic of the group as a whole which determine that trait, we shall expect the manifestations of that trait by the group of which he is a sample to be distributed in Form *A*.

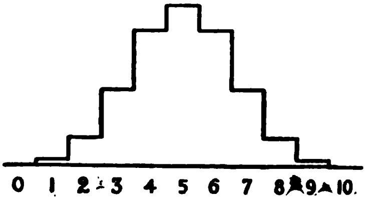

FIG. 32.

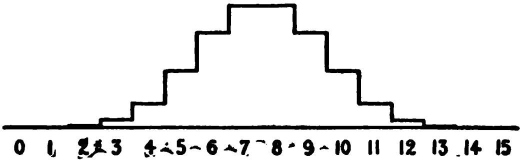

FIG. 33.

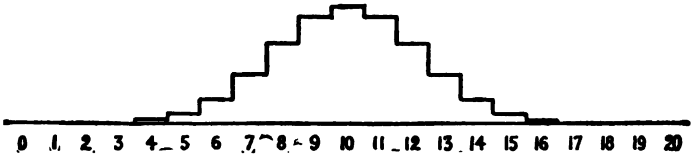

FIG. 34.

The clause 'so that any combination is as likely to occur as another' and its synonymous phrase 'a chance combination from amongst' need some explanation. They refer to the fact that the causes must be independent of each other if the distribution of the trait is to be of Form *A*. The need of this condition will be apparent from the facts of the next section.

§ 18. *The Effect of Dependence and Unequal Potency*

Suppose that in our previous case of John's Christmas money the six causes *a, b, c, d, e* and *f* were as before, except that no action was barred out, and that if *a* acted *b* and *c* must also, and *d, e* and *f* could not; while if *d* acted *e* and *f* must, but *a, b* and *c* could not. Imagine, for instance, that it was agreed to take turns in preventing a penniless Christmas; that the father agreed to give his dollar if the mother and sister would always join with him and the grandfather, grandmother and brother would keep their money to themselves, while the grandfather agreed to give his dollar upon the condition that he be joined by grandmother and brother and that father, mother and sister refrain. The condition of affairs then could only be *abc* or *def* instead of the range of possibility of the illustration in its first form. Although there are six causes, the result is as if there were only one, and that always operative.

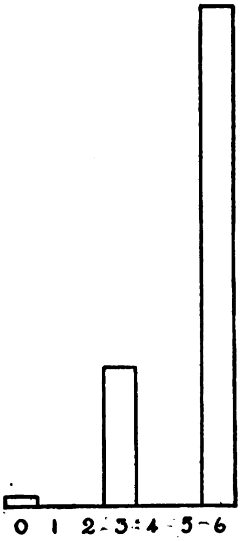

FIG. 35.

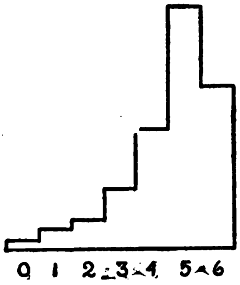

FIG. 36.

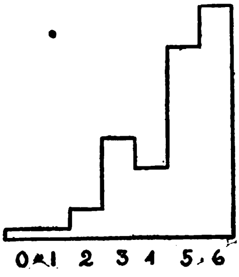

FIG. 37.

Suppose the presence of *a* or *b* or *c* to always cause that of the other two of the three, and similarly for the presence of *d, e* or *f*. This means that whenever cause *a* appears it adds to itself *b* and *c*, whenever *b* appears it adds to itself *a* and *c*, and so on. Every condition in Table 16 with *a* or *b* or *c* in it must then become *abc*;

every condition with *d* or *e* or *f* must become *def;* every condition with one from the *abc* and one from the *def* group must become *abcdef.* Thus the condition of affairs would be, instead of that in Table 16, the following: *no action,* 1; *abc,* 7; *def,* 7; *abcdef,* 49.

The distribution would then be (as shown in Fig. 35):

Quantity: Dollars	Frequency
0	1
3	14
6	49

Suppose the presence of *a* to imply always that of *c, d, e* and *f,* the presence of *b* to imply always that of *d, e* and *f,* the presence of *c* to imply that of *e* and *f,* and the presence of *d* that of *f.* The distribution would be (as shown in Fig. 36):

Quantity: Dollars	Frequency
0	1
1	2
2	3
3	6
4	12
5	24
6	16

Suppose the presence of *a* or *b* or *c* implies the other two of the three, and that the presence of *e* implies that of *f,* and *vice versa.* The distribution will be (as shown in Fig. 37):

Quantity: Dollars	Frequency
0	1
1	1
2	3
3	10
4	7
5	19
6	23

It is clear then that the interdependence of the causes determining the quantity of a trait may cause all sorts of departures from the normal type of distribution, skewnesses and multimodal conditions, etc.; may, in less technical terms, cause the amounts of it appearing in an individual's different records or in the different individuals of a group to vary in all sorts of ways. In the illustration only simple and total dependencies were considered. Complex and partial dependencies would complicate the results to a well-nigh endless extent.

It should, however, be noted that if the causes are numerous and their interdependences of a random, hit-or-miss character, their combined action may be practically identical with that of totally independent causes. Thus, to continue with the same illustration, if there were five hundred relatives they might plan together in groups on various ways to give or withhold, and yet the final resultant, the probable total of John's Christmas income, might show no considerable differences from the total in case they had all acted independently.

The same infinite variety in the form of distribution may be brought about by inequality in the magnitude of the causes. Of this the reader may best convince himself by so varying the magnitude of *a*, *b*, *c*, *d*, *e* and *f* in Table 16. For instance let $a = 10$, $b = 5$, and let *c*, *d*, *e* and *f* each equal 1. Then we have, as shown in Fig. 38,

Quantity Dollars	Frequency
0	1
1	4
2	6
3	4
4	1
5	1
6	4
7	6
8	4
9	1
10	1
11	4
12	6
13	4
14	1
15	1
16	4
17	6
18	4
19	1

Here again, however, with many causes and with a not too great variation in their amounts, the resulting distribution may approximate closely to Form *A*.

Finally, it should be remembered that the illustration taken is untrue to the common conditions of life in one respect. For these show us, not a group of causes, a chance combination from which determines the event, but such a group acting *together with some constant cause* or *set of causes*. Stature, strength, memory, wage-earning capacity, are due to certain constant causes which always act on all, plus a group, the action of which may be regarded in the mathematical fashion of this chapter. The addition of such a constant set of causes does not, of course, alter the form of distribution in the least, but simply adds the same amount to all its quantities, pushes them all ahead on the scale. In our illustration the *a*'s, *b*'s, *c*'s, etc., might more properly be the amounts which different friends might or might not give in addition to minimum sums, k, k_1, k_2, etc., which they always give, or be the gifts of some friends, who could not be counted on, superadded to a set of inevitable gifts x, y, z, etc., from a few.

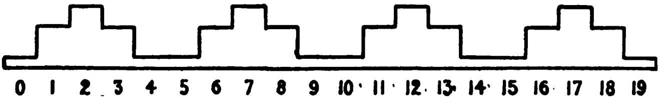

FIG. 38.

The commoner method of describing the type of causation resulting in the probability surface of frequency of the amount of a trait starts with the presupposition that a certain amount tends to be and considers the causes as increasing or decreasing this. It is also common to use the frequencies, not of amounts of some continuous quantity, but of different proportions of black and white, or the like, in a chance draw of balls. The principles involved are precisely the same as those which have appeared in the more readily understood cases used here.

I have so far tried especially to show how the cooperation of a number of causes, each of which has a given likelihood of acting, may produce in the trait due to them a distribution of Form *A*. Incidentally, it has been noted that in general the form of distribution of any variable trait is due to the number of causes that influence its amount, their magnitude and their interrelations.

The form of distribution then is purely a secondary result of a trait's causation. There is no typical form or true form. There is nothing arbitrary or mysterious about variability which makes the so-called normal type of distribution a necessity, or any more rational than any other sort or even any more to be expected on *a priori* grounds. Nature does not abhor irregular distributions.

On *a priori* grounds, indeed, the probability curve distribution would be exactly shown in any actual trait only by chance. For only by chance would the necessary conditions as to causation be fulfilled. And in point of fact, as the reader will frequently be told by the adjective "approximate," the exact probability curve distribution does not appear in the facts or give signs of being at the bottom of the facts of mental life. The common occurrence of distributions approaching it is due, not to any wonderful tendency of a group of cooperating causes to act so as to mimic the combinations of mathematical quantities equal and equally probable, but to the fact that many traits in human life are due to certain causes plus many occasional causes largely unrelated, small in amount in comparison with the constant causes and of the same order of magnitude among themselves.

It is the folly of the ignoramus in statistics to neglect the application of the algebraic laws of combinations to variable phenomena; it would be the folly of the pedant to try to bend all the variety of nature into conformity with the one particular case of the frequency of combinations which results in Form A for the total distribution.[1]

PROBLEMS

26. Suppose that, in Table 16, $a = 6, b = 5, c = 4, d = 3, e = 2$, and $f = 1$. Draw the resulting form of distribution for a chance selection from the combinations possible.[2]

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[1] It is a question whether students of mental measurement should not from the beginning be taught to put the normal probability distribution in its proper place as simply one amongst an endless number of possible distributions, each and all due to and explainable by the nature of the causes determining the variations in the trait. The frequency of the occurrence of distributions somewhat like it could then be explained by a *vera causa*, the frequency of certain sorts of causation. On general principles this seems desirable, but in order to make for the student connections with the common discussions of statistical theory and practise and with the concrete work that has been done with mental measurements, I have compromised and, to some extent, subordinated the general *rationale* of the form of distribution to the explanation of the probability curve type.

[2] Remember that *ab*, *ade*, and the other entries of Table 16 mean that the effect of a and that of b are added; that the effects of a, d and e are added, and similarly for every combination.

27. Suppose that, in Table 16, $a = 10$, $b = 10$, $c = 10$, $d = 1$, $e = 1$, and $f = 1$. Draw the distribution as in 26.

28. Suppose that, in Table 16, the presence of a and b together implied that of c and that the presence of d and e together implied that of f. Draw the resulting distribution, if a, b, c, d, e and f are, in each case, equal to 1.

CHAPTER VII

THE MEASUREMENT OF A GROUP

§ 19. *The Use of Measures of Individuals to Obtain Measures of Groups*

THE sciences of human nature often use measures of individuals only in order to get measures of groups. Not John Smith's spelling ability, but that of all fifth grade boys taught by a certain method; not *A*'s delicacy of discrimination of weight, but that of all men; not *B*'s wage, but that of all railroad engineers during a certain period; not the number of *C*'s children, but the productivity of the English race as a whole; not individuals, but groups, are often the facts to be measured, compared and argued about.

Variations amongst the Individuals of a Group.—The customary expression of a trait or ability in a group is its average, and the use of an average here, as before, points to the variability of the fact. We do not seek the average law of gravity, or the average ratio of amount of oxygen to amount of hydrogen in an atom of water, or the average velocity of sound. It is because of the unlikeness, the variability, of even the most similar individuals in even the most constant qualities that we are forced to use averages at all.

An average no more represents the different members of a group than it did the different measures of a trait in a single individual. The thing, fact *A* in the individuals of group *X*, is a variable quantity and is measured only by a list of the different degrees of the trait found in all the individuals of the group, with a statement of the number of times each appears. A table of frequencies or surface of frequency will be the adequate measure here, as before. The measure of a fact in a group is its total distribution, and this total distribution is simply all the separate measures of the individuals making up the group.

The measure taken for each individual may be his average, or his most frequent ability, or highest ability shown, or lowest ability shown, or ability exceeded in 50 per cent. of his trials, or ability exceeded in 70 per cent. of his trials, or variability or any other characteristic of "individual in group *X*."

Means of Measuring the Central Tendency and the Variability of a Group.—The determinations of the central tendency and variability of a measure of a group are made in just the same way as in the case of a measure of an individual, and the different measures of them have here the same characteristics. The formal and mathematical problem is identical whether we have varying ✓ records of one individual or varying individuals of one group.

Instead of "central tendency of the different measures of one individual in respect to some trait" we have "central tendency of the different individuals of one group in respect to some trait." Instead of "variability of the different measures of one individual in respect to some trait" we have "variability of the different individuals of one group in respect to some trait."

As in the case of individual measures, it is a safe rule never to replace the total distribution by any partial expressions of it until it is necessary. As in the case of an individual measured, the distributions may conceivably take all sorts of forms and be quite unrepresentable by any simple arithmetical constants.

The Effect of Inadequate Measures of the Individuals.—An accurate representation of the central tendency of a group may be had from very inadequate measurements of the individuals in it—for instance, from records of only one or two of the varying scores of each individual. The reason is, of course, that, the errors being chance errors, the too high rating of individual *A* is counterbalanced by the too low rating of *B*, and so on; so that the central tendency for the group as a whole is substantially as it would have been had each individual in it been measured a hundred or more times. Thus, the first column of frequencies in Table 18 gives the distribution of the abilities of a hundred individuals, in a test of sensory acuity, twenty records from each individual being used. The second column of frequencies gives the distribution when only four records taken at random from the twenty, were used. The central tendency computed from the second column differs from that computed from the first by only one fourth of one unit of the scale, or about one per cent. of the total range.

TABLE 18

AVERAGE ERROR IN DRAWING A LINE TO EQUAL A 100-MM. LINE

A = averages calculated from 20 trials for each individual.
B = averages calculated from 4 trials.

Quantity: Tenths of Millimeters		Frequencies	
		A	B
−100 to −120		1	1
− 80 to −100		3	4
− 60 to − 80		7	5
− 40 to − 60		12	5
− 20 to − 40		17	18
0 to − 20		18	24
0 to + 20		13	17
+ 20 to + 40		17	15
+ 40 to + 60		5	6
+ 60 to + 80		4	1
+ 80 to +100		3	3
+100 to +120			0
+120 to +140			1
	Averages	−.72 mm.	−.46 mm.
	Medians	−.889 mm.	−.584 mm.

The effect of inadequacy of the measures of the individuals in it upon the variability of the group, is, on the other hand, to produce an error which acts in the long run to make the A.D., S.D., Q, or any other measure of the dispersion of the individual members about their central tendency, *too large*. That this is the case can be easily seen by comparing the dispersion of the measures of the group a–j got by taking for each individual in Table 19 the average of his ten scores, with the dispersion got by taking for each individual one or two scores at random. That it must be the case can be inferred from the following: Let A, B, C, D, etc., be the adequate measures of certain individuals in a certain respect. Let Z be the dispersion of A, B, C, etc., around their central tendency. Let a_1, a_2, a_3, etc., be the separate scores from which the adequate measure A is derived. Let b_1, b_2, b_3, etc., have the same relation to B; let c_1, c_2, c_3, etc., have the same relation to C; and so on. Let w_1 be the dispersion of $a_1 \ldots a_n$ around A; let w_2 be the dispersion of $b_1 \ldots b_n$ around B; and so on.

Then V, the dispersion of the individuals around their central tendency when measured inadequately by say a_1 and a_2, b_1 and b_2, c_1 and c_2, etc., may be considered as the result of the combination of the causes producing Z and those producing w_1, w_2, w_3, etc. Since w_1, w_2, w_3, etc., are mutually uncorrelated and are uncorrelated with Z, the result of the combination of causes is to make V greater than Z.

TABLE 19

THE SCORES OF TEN INDIVIDUALS EACH IN TWELVE INDEPENDENT TESTS OF THE TRAIT IN QUESTION, AND THE AVERAGE OF THE TWELVE FOR EACH INDIVIDUAL

	1st Score	2d Score	3d Score	4th Score	5th Score	6th Score	7th Score	8th Score	9th Score	10th Score	11th Score	12th Score	Average of All 12 Scores
a	22	9	31	18	19	21	20	20	24	27	13	16	20
b	18	20	24	21	33	15	23	11	22	29	22	26	22
c	26	14	24	18	23	36	25	32	27	25	21	29	25
d	19	37	34	28	23	30	30	41	26	32	29	31	30
e	31	27	20	31	29	42	38	30	32	35	24	33	31
f	33	29	25	43	31	36	21	32	30	28	32	34	32
g	35	34	36	39	24	35	46	42	31	33	28	37	35
h	36	26	39	44	48	37	35	38	30	33	41	37	37
i	37	40	49	38	36	45	39	42	31	38	27	34	38
j	41	47	29	38	33	51	42	36	39	40	40	44	40

§ 20. *The Extent to Which the Surface of Frequency of* "Fact T in the Case of the Different Individuals in Group $a \ldots n$," *Approximates Form A, the So-called* "*Normal*" *Form*

The differences amongst individuals in certain groups, with which students of mental science have to deal do, in the case of most anatomical traits, of very many physiological traits, of many mental traits and of at least some institutional and social traits, are such as to produce in the measurement of the group an approximation toward a unimodal distribution the variability of which is of such a nature as to justify one in regarding the members of the group as representatives clustering about a type, departures from which show a certain regularity. In other words, distribution is often unimodal, the statistical average or mode very often represents a real central tendency or type, and, the departures from it occurring in an orderly way, one or two figures can often represent the real clustering of individuals about a type.

In particular there is found often a form of distribution (1) approximating the symmetrical, with its mode approximately at the average, so that both are nearly coincident with the median, and (2) characterized approximately by a slow decrease in frequency for a certain distance above and below the mode, a more rapid decrease from then on for a way, and finally a slower decrease until the limits are reached. This description the reader will recognize as the description of a distribution of approximately Form *A*, that of a quantity determined by the action of a large number of independent causes equal in amount; in other words, that of the surface bounded by the probability curve.

In so far as this particular uniformity in distributions does exist, we are freed from the necessity of devising a separate means of quantitative expression for each group measurement studied, and permitted to express it at least approximately in two figures, one telling the general tendency or type, the other the variability, the form being assumed to be approximately Form *A*.

I have represented graphically in the following pages distributions of a number of anatomical, physiological, mental, social and institutional facts, drawing them so that a rough comparison with the surface of frequency of the probability integral can be made in each case.[3] The examination of these will give a concrete and reasonably accurate notion of the frequency with which the measurement of a group is again and again approximately the same statistical problem.

In these diagrams (Figs. 39 to 65) the continuous lines enclose the surface of frequency of the fact in question. The dotted lines give approximately the surface which would be found if the distribution of the trait followed Form *A*, the probability surface. Where the actual distribution obviously is not even approximately of this form, the dotted lines are omitted. The exact nature of the trait, the number of individuals and the source of the data in each case are given in the list that follows. When no source is stated the author is responsible for the original data. Unfortunately, the equality of the steps taken as equal in the scales by which the facts of Figs. 49, 50 and 52 were measured, is far from certain. Consequently the diagrams may not represent the true form of distribution in these cases.

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[3] In the eight years since the first edition of this book appeared the practise recommended in it—of reporting the entire distribution of any variable fact instead of merely its average or average and variability—has become the customary one with many workers, and a very great number of further illustrations could now be printed.

FIG. 39. Height of American adult men. In inches. N (number of cases) = 25,878. Drawn from the table given by Karl Pearson on page 385 of Vol. 186A of the *Philosophical Transactions of the Royal Society of London.* He quotes from J. H. Baxter, Medical Statistics of the Provost Marshal's General Bureau.

FIG. 40. Weight of English adult men. In pounds. $N = 5{,}552$. Drawn from the table given in C. Roberts' "Manual of Anthropometry"; appendix.

FIG. 41. Cephalic Index (ratio of width to length of head) of modern Alt-Bayerische skulls. $N = 900$. Drawn from the table given by Karl Pearson in "The Chances of Death."

FIG. 42. Length of male infants at birth. In inches. $N = 451$. Source the same as for Fig. 40.

FIG. 43. Girth of chest, empty, of English army recruits. In inches. $N = 675$. Source the same as for Fig. 40.

FIG. 44. Strength of arm pull. English adult men. Pull exerted as in drawing a bow. In pounds. $N = 1{,}497$. Source the same as for Fig. 40.

FIG. 45. Body temperature at the mouth in American women. $N = 158$. I am indebted for the original measures to Professor T. D. Wood, of Teachers College.

FIG. 46. Heart rate (after vigorous exercise) in American students, young men 16 to 20. Number of beats per 60 seconds. $N = 312$. I am indebted for the original measures to Dr. G. L. Meylan, of Columbia University.

FIG. 47. Reaction time of American college freshmen. Thousandths of a second. $N = 252$. I am indebted for the original measures to Dr. Clark Wissler, of the American Museum of Natural History.

FIG. 48. Memory span for digits in American women students. Number of digits correctly written and correctly placed. $N = 123$.

FIG. 49. Efficiency in perception of 12.5-year-old boys. Number of *A*'s marked in 60 seconds on a sheet of 13 lines of capital letters (see sample below). $N = 312$.

OYKFIUDBHTAGDAACDIXAMRPAGQZTAACVAOWLYXWABBTHJJAN
EEFAAMEAACBSVSKALLPHANRNPKAZFYRQAQEAXJUDFOIMWZSA
UCGVAOABMAYDYAAZJDALJACINEVBGAOFHARPVEJCTQZAPJLEIQ
WNAHRBUIAS

FIG. 50 Efficiency in controlled association of 12.5-year-olds. Number of correct minus number of incorrect opposites of the following words written in 60 seconds: Good, outside, quick, tall, big, loud, white, light, happy, false, like, rich, sick, glad, thin, empty, war, many, above, friend. $N = 239$.

FIG. 51 Accuracy of estimation of length in girls 13 to 15 years old.[4] Average variable error, in millimeters, in 30 attempts to draw a line equal to a 100-mm. line seen. $N = 153$.

FIG. 52. Efficiency in complex perception of 12.5-year-old boys. Number of words containing *a* and *t* marked in 120 seconds in a sheet of words (see sample below). $N = 312$.

Dire tengo antipatia senores; esto seria necedad, porque hombre vale siempre tanto como otro hombre. Todas clases hombres merito; resumidas cuentas, sulpa suya vizxonde; pero dire sobrina puede contar dote viente cinco duros menos, tengo apartado; pardiez tamado trabajo atesorar-los para enriquecer-estrano.

FIG. 53. Ratio of attendance to enrollment in public schools of cities and towns of over 8,000 inhabitants in Ohio, Indiana, Illinois and Iowa. $N = 115$.

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[4] The 13-, 14- and 15-year old girls did not differ as groups.

FIG. 54. Wages of cotton operatives (in shillings per week). N is large, but not given. The data are taken from Bowley's "Elements of Statistics," p. 96.

FIG. 55. Age of graduation from American colleges. Men only taken. $N = 1{,}213$.

FIG. 56. Cost per pupil of public school education in American cities of over 8,000 inhabitants. The cost is here taken per pupil actually present throughout the year. That is, the cost per pupil equals amount spent divided by average attendance. In dollars. $N = 465$. The amounts and average attendances are those given in the Report of the U. S. Commissioner of Education for 1901.

FIG. 57. Wages of American workingmen per day. In cents. $N = 5{,}123$. The data are taken from Bowley's "Elements of Statistics," p. 120. He quotes them from a U. S. Senate report.

FIG. 58. Fig. 39 with a coarser grouping.

FIG. 59. Ratio of attendance to enrollment in public schools of American cities of over 8,000 inhabitants. $N = 545$.

FIG. 60. Incomes of American colleges for men and for both sexes. The five per cent. who in the year taken had incomes of over $150,000 are omitted. In thousands of dollars. $N = 438$.

FIG. 61. Age at marriage of gifted American men. $N = 744$.

FIG. 62. Frequency of divorces in different years after marriage. The cases after twenty years are undistributed by the compiler and are here given a probable distribution. $N = 109{,}960$. The data were taken from Karl Pearson's table, *Phil. Trans. of the Royal Society*, Vol. 186A, p. 395. He in turn quotes them from W. F. Wilcox, "The Divorce Problem."

FIG. 63. Size of New England families, 1725–1800. The number of children born to women during twenty years or over of married life. $N = 163$.

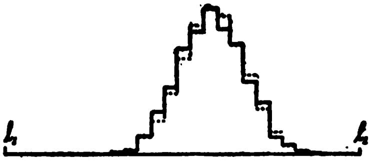
FIG. 64. Infant mortality in cities and towns of England and Wales. Number of deaths per 1,000 births. $N = 112$. Arranged from data given by Miss Clara Collet in the *Journal of the Royal Statistical Society*, June, 1898.

FIG. 65. Frequency of death at different ages. After Karl Pearson, "Chances of Death," Vol. I., p. 27. N is very large.

In figures 39 to 65, the limits to which the surface of frequency extends are shown by short vertical lines in those cases where the length of the columns of which it is composed is so small as to be unnoticeable. See, for instance, l_1 and l_2 in Fig. 39.

It appears from the illustrations given here and from the larger group from which they are a selection, that when the distribution of the individuals in a group is around one type, the form of the clustering is more often like the normal form than like any other *one* form. Consequently when, in ignorance of the actual form, some form has to be assumed, form A or a modification of it is the best one to assume. On the other hand, the approximations are so imperfect that the assumption, though the best single one, is essentially unsafe.

It is, of course, not desirable to have to make any assumption about the form of a surface of frequency. Whoever reported the central tendency and variability should have reported the entire



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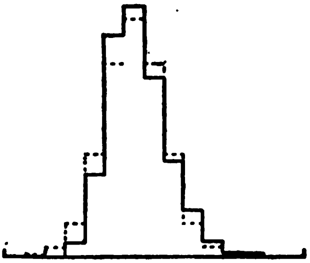

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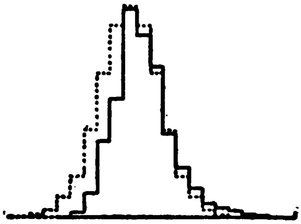

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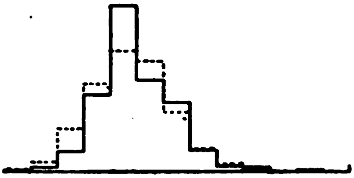

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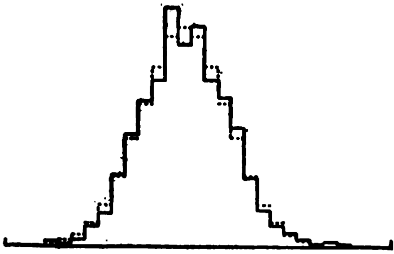

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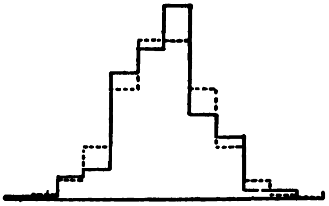

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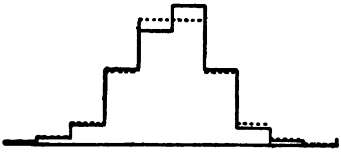

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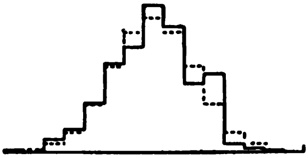

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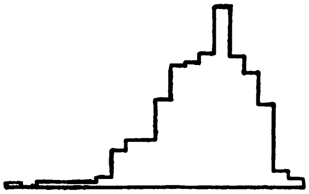
FIG. 39. Height of men.
FIG. 40. Weight of men.
FIG. 41. Cephalic index.
FIG. 42. Length of infants.
FIG. 43. Girth of chest.
FIG. 44. Strength of arm pull.
FIG. 45. Body temperature.
FIG. 46. Heart rate after exercise.



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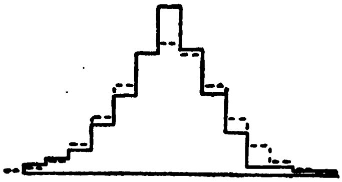

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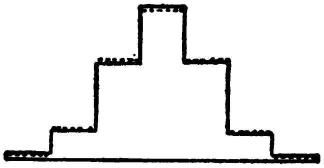

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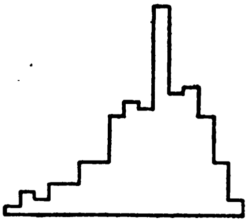

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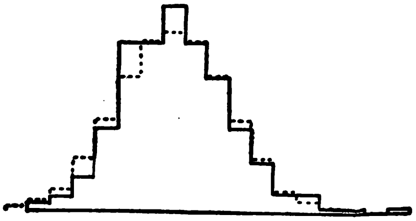

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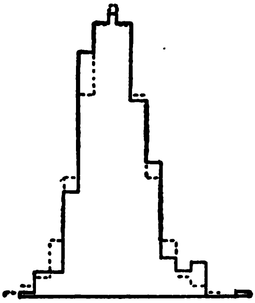

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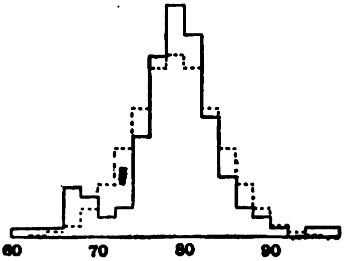
FIG. 47. Reaction time.
FIG. 48. Memory span for digits.
FIG. 49. Efficiency in perception of *A*'s.
FIG. 50. Efficiency in association of ideas.
FIG. 51. Accuracy of estimation of length.
FIG. 52. Efficiency in perception of words.



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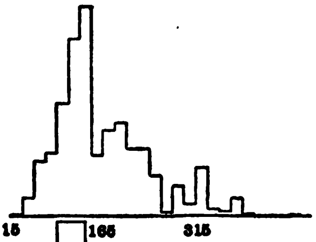

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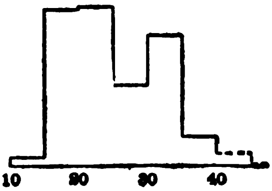

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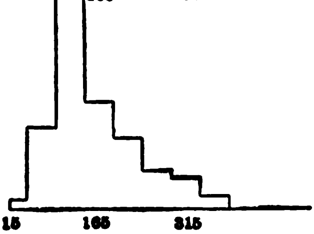

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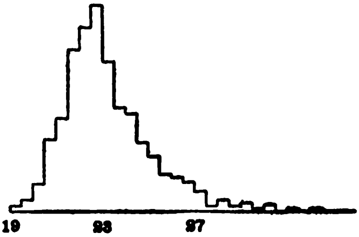

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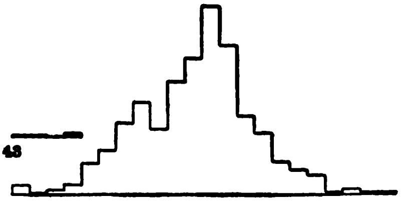

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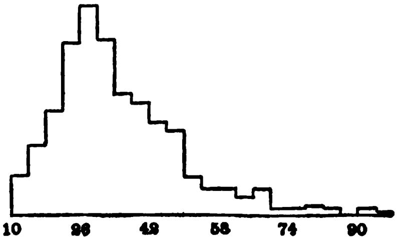

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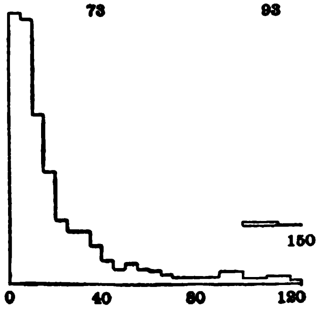

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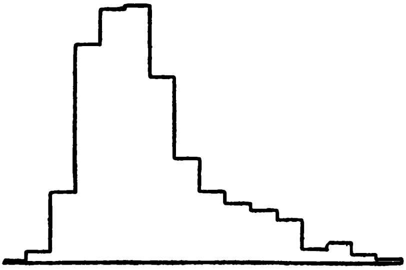
FIG. 53. Ratio of school attendance to enrollment.
FIG. 54. Wages of cotton operatives.
FIG. 55. Age of graduation from college.
FIG. 56. Cost per pupil of education.
FIG. 57. Wages of American workingmen.
FIG. 58. Wages of American workingmen.
FIG. 59. Ratio of school attendance to enrollment.
FIG. 60. Incomes of colleges.



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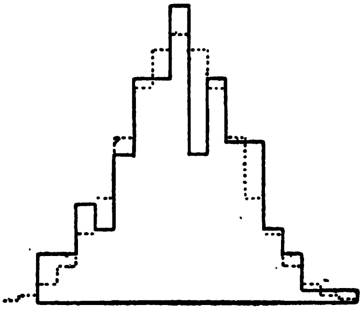

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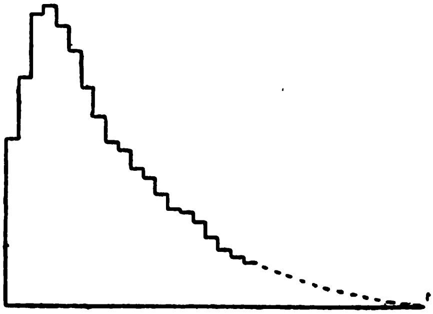

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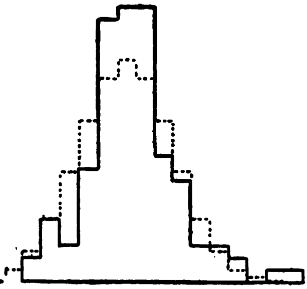

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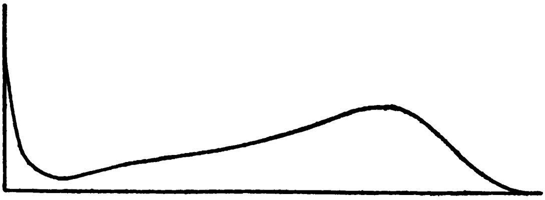

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FIG. 61. Age of marriage of gifted men.

FIG. 62. Frequency of divorces at different dates after marriage.

FIG. 63. Size of New England families.

FIG. 64. Infant mortality.

FIG. 65. Frequency of death at different ages.

table of frequencies or a graphic representation of them or a clear statement of their geometrical form. But until about 1900 such full reports of variable facts were rare in the literature of psychology, sociology, education, or the other mental and social sciences, and they are still far from universal. Hence many of the measurements that exist have to be interpreted by some more or less speculative supposition about the form of the surface of frequency if they are to be used in detail at all. It is also often necessary to make an assumption concerning the form of distribution in the case of traits where equality in the units of the scale is dubious.

§ 21. *The Interpretation of Divergences from Form A in the Distribution of a Group*

Such Divergences May Be Significant.—The form of distribution for any group deserves careful study.

For instance, if in a measure of the scholarship of men one obtained a distribution like that represented in the upper diagram of Fig. 66, it might appear reasonable to say that intellect was distributed in a very irregular manner and in such a way that there were no grades very far below the commonest condition, but that grades above it existed over such a range that the highest ranking person was ten times as far above the mode as the lowest ranking person was below it, and that the grades up near the highest were more common than those a little nearer the mode. Further consideration, however, might show that the infrequency of low grades was due to the fact that in our measurements we had tested only the better classes—had selected against the idiots, illiterates and incompetents; and that the apparently greater frequency of very high grades than of moderately high grades was due to our having measured some thousands of individuals from the better classes together with a few hundred expert scholars. Scholarship in general might really be distributed normally as shown in the lower diagram of Fig. 66, and our result be due to the influence of selection and of mixing two species, untrained and trained men. On the other hand, if one obtained for scholarship a normal distribution, one could not be sure that in the natural group, men, scholarship was normally distributed, unless these same factors of elimination and mixture were excluded. For example, if one got a normal distribution from measuring 13-year-old boys in the next to the last grammar-school grade, he could be practically sure that for all 13-year-old boys the distribution would *not* be normal. The duller 13-year-old boys would not have reached that grade and the very bright ones would often have passed it. The actual distribution may be in part the result of the mixture of species or of selection.

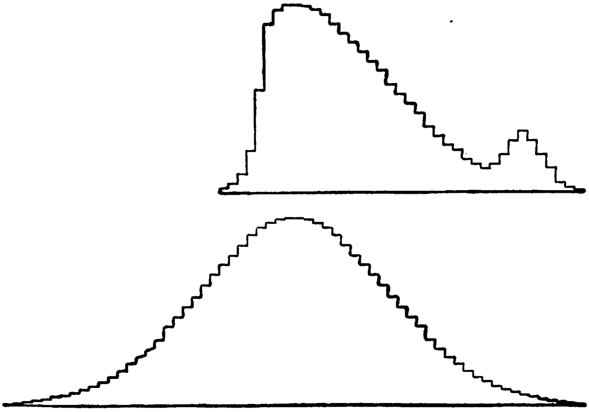
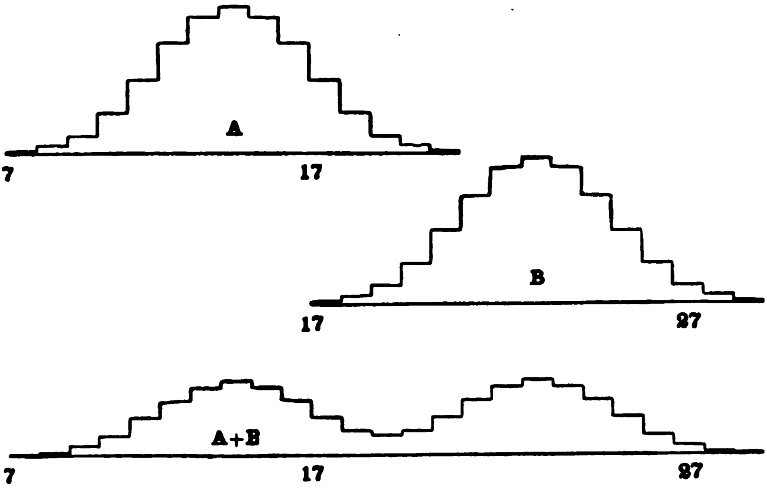
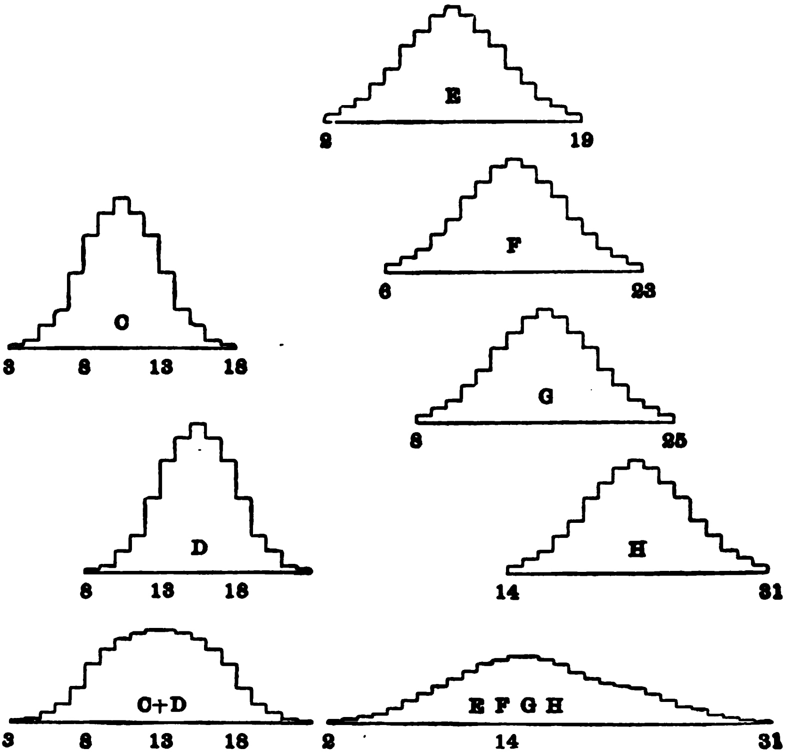


FIG. 66. An irregular distribution (upper diagram) possibly due to artificial elimination and mixture in the course of the measurements, the true fact being that shown by the lower diagram.

Homogeneous and Mixed Groups.—Homogeneity is in general not an absolute, but a relative, quality. A group of animals is homogeneous compared with a group of animals and plants mixed. A group of human beings is homogeneous compared with a group of men, dogs, worms and fishes. A group of college graduates is homogeneous compared with a group of college graduates, illiterates and idiots. Utter homogeneity would equal identity. We commonly mean by the homogeneity of any group with respect to any trait, such likeness amongst its members, with respect to the forces producing the trait, that there is no reason for separating them into several groups rather than leaving them in one. Thus the group 'a species' of the zoologist or botanist is homogeneous with respect to its anatomy. Thus the group 'children of the same race, sex and age' is probably homogeneous with respect to the trait 'maturity.' Thus the group 'wages of unskilled laborers under the same conditions of work and cost of living' is homogeneous to the economist.


A
7
17
B
17
27
A+B
7
17
27


E
2
19
F
6
23
C
3
8
13
18
G
8
25
D
8
13
18
H
14
31
C+D
3
8
13
18
E F G H
2
14
31

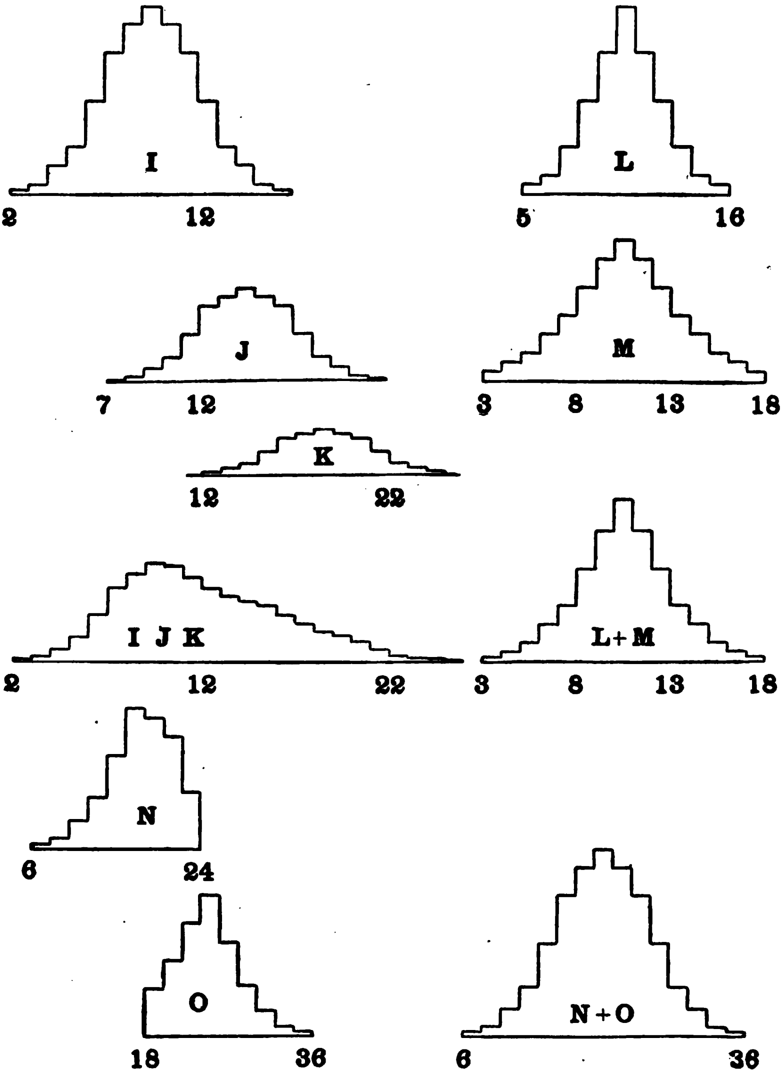


FIG. 67. Showing six cases of the influence of combination upon the form of distribution, viz:

Two normal distributions, *A* and *B*, when combined, give a markedly bimodal distribution.

Two normal distributions, *C* and *D*, when combined, give a flattened distribution.

Four normal distributions, *E*, *F*, *G* and *H*, when combined, give a flattened and positively skewed distribution.

Three normal distributions, *I*, *J* and *K*, when combined, give a markedly skewed distribution.

Two distributions, *L* and *M*, of identical mode but differing variability, give, when combined, a form midway between the two.

Two distributions, *N* and *O*, one positively and the other negatively skewed, give, when combined, a normal distribution.

The effect on the distribution of a trait of putting together groups different as groups with respect to the trait can be seen from the diagrams of Fig. 67.

It is obvious, in general, that given any form of distribution, it might be accounted for, so far as the bare fact of its existence went, by any one of a practically infinite number of different compoundings of groups. The mere form of distribution does not itself tell. Recourse must be had to a study of the real facts about the group.

I shall consider further only the case of the compounding of two or more groups, each of which by itself shows approximately normal distribution, which differ in respect to the amount of the trait. It is clear from the diagrams that the result on the form of distribution of the total group will be multimodality or a flattening of the top of the surface of frequency at some point. If one has reason to believe that the trait he is studying would in a homogeneous group show normal distribution, the existence of such multimodality or flattening may properly lead him to suspect the mixture of two groups or species and to examine the cases with a view to separating them into more homogeneous groups.

One special case of importance is that where the total group is a compound of a very large number of groups so differing that their central tendencies form approximately an arithmetical series. Such total groups would be, for instance, measurements of children eight to twelve years of age in some physical or mental trait subject to growth, or of teachers' salaries over a period of years during which there was a steady rise in values.

The Effect of Selection and Elimination.—Only very infrequently does the measurement of any trait in a group include all the members of a group. It is, on the contrary, the result of measurements of relatively few sample individuals. These represent the group as a whole justly only in so far as they include the same percentage of each grade of ability in the group.

In general, it can easily be shown that by the right combinations of selections from a group, a group with any form of distribution can be derived, no matter what the form of distribution of the trait in the original group was.

Selection may occur (1) as a result of natural forces upon a group, or (2) as the result of unproportional sampling by the measurer. The group, living human beings 40 years old, is thus the selection by natural forces from the group, all human beings born 40 years ago, a selection, to some extent at least, of the physically more vigorous, morally less murderous, and so on. The group, seventeen-year-old boys measured in school, is a selection from all boys seventeen years old, due to the measurer's willingness to take boys not absolutely at random, but as found conveniently. The selection is, to some extent at least, of the more ambitious and gifted intellectually.

Consequently an examination of the form of distribution with an eye to evidence of selection is often very profitable. The influence of nature in changing the distribution of a trait in a group by selecting for survival on the basis of the trait's amount is one of the most important topics for science, and the influence of circumstances in providing the student with a set of selected samples the distribution of which is unlike that of the total group the student takes them to represent, is an important cause of fallacies in the mental and social sciences.

Although any form of frequency surface may be derived from any other by the proper method of selection of cases, and although, consequently, from the actual form of a surface of frequency nothing can be concluded concerning the group from which it represents a selection unless the method of selection is known, yet certain appearances may well serve to awaken suspicion and lead the student to investigate the measurements. In particular, skewness is so often connected with picking for study extreme cases of a group, which as a whole would give an approximately normal distribution, that it is certainly advisable always, when confronted by a group measure showing skew distribution, to ascertain whether the group is not a partial picking from a normally distributed total group.

On the whole it may be said that the interpretation of the form of distribution of a group is a most valuable element in statistical procedure, if one does not expect too much from it. To the student who is acquainted with the nature and meaning of the trait measured and with many of the characteristics of the group in which it is measured, the form of the distribution may suggest other characteristics of the group—in particular, the characteristic of being a mixture requiring analysis into separate groups, or a selection not representative of the total which it pretends to sample fairly. These suggestions can then be tested.

The form of distribution, taken alone, does not, however, demonstrate anything concerning individual differences in general, or the mixed or selected composition of the group, or anything else beyond the mere fact that such and such individuals gave such and such measures showing such and such differences in respect to the trait in question. The form of distribution should then be examined with intelligent consideration of all the facts known about the units of measure, the trait and the group.

CHAPTER VIII

THE TRANSMUTATION OF MEASURES BY RELATIVE POSITION INTO MEASURES IN UNITS OF AMOUNT

§ 22. *Transmutation by Means of Knowledge of the Form of the Distribution*

FOR the sake of simplicity, only the case of individuals measured by their relative position in a group will be discussed in this chapter. The theory and technique described apply equally to any series of facts ranked in order for their amounts of any one trait.

If a group of individuals are ranged in order according to the amounts which they severally possess of a trait, we can, even when ignorant of what the amounts are for each and all of the individuals, assign to each the amount of his deviation from the average, provided the form of the group's distribution is known. For instance, let 100 boys rank with respect to scholarship as shown below, and let the form of distribution be that of Fig. 68.

100 BOYS—a, b, c, ETC.—RANKED BY RELATIVE POSITION

a	is the highest ranking boy.
b, c, d	are next in rank and are rated equal.
e, f, g, h, i, j	are next in rank and are rated equal
k, l, m, n, o, p, q, r, s, t	are next in rank and are rated equal.
u, v, w, x, y, z, *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*	are next in rank and are rated equal.
j, *k*, *l*, *m*, *n*, *o*, *p*, *q*, *r*, *s*, *t*, *u*, *v*, *w*, *x*, *y*, *z*	are next in rank and are rated equal.
A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S	are next in rank and are rated equal.
T, U, V, W, X, Y, Z, α, β, γ, δ, ϵ, ξ, η	are next in rank and are rated equal.
θ, ι, κ, λ, μ, ν, ξ, ο	are next in rank and are rated equal.
π, ρ, σ, τ	are next in rank and are rated equal.
υ, φ, χ	are next in rank and are rated equal.

If we build up approximately the surface of Fig. 68 by a series of forty rectangles of equal base, the result is Fig. 69. This, the reader should observe, is done graphically by dividing the base line arbitrarily into forty equal parts, and by erecting a rectangle on each division of the base line—of such height that the mid-point of its top is at the intersection of its top with the bounding line of the distribution of Fig. 68. There is no reason for the division of the base into forty rather than 60, or 70, or 120, equal parts.

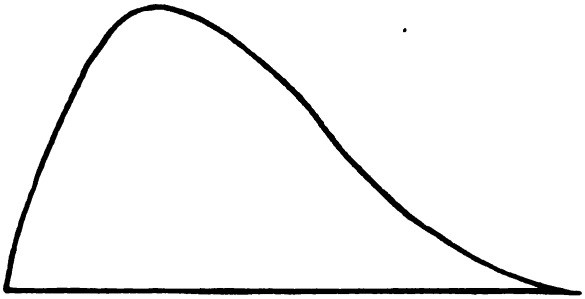


FIG. 68.

Call the distance of the low extreme from the absolute zero, A; and call the length of base of each of the rectangles, K. Then the upper extreme is at $A + 40K$, and the relative frequencies for the fortieths of the range—that is, the relative heights of the forty rectangles—are as noted in Table 20, the total area being taken to be 1,680. These relative frequencies can, of course, be reckoned on the basis of any arbitrary value for the total area. There is no reason, save convenience, for assuming the area to be 1,680 rather than 2, 16,000, 1,820, or any other number. The 1,680 was an accident of the particular scale used to measure the area of Fig. 69. If the reader will construct an approximation made up of 60, or 80, or 100, rectangles, and call the total area 1, 2, 500, 10,000 or any other number, he will still get the same final values for the distances of a, b, c, d, etc., from any defined point along the base-line, within

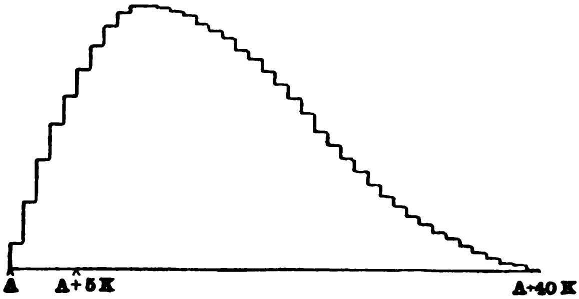


FIG. 69.

the range of the distribution, in terms of any defined feature of the distribution, say its range, A.D., σ (S.D.), or Q.

TABLE 20

Quantity	Frequency
A to $A + K$	7
$A + K$ to $A + 2K$	20.5
$A + 2K$ to $A + 3K$	33
$A + 3K$ to $A + 4K$	44
$A + 4K$ to $A + 5K$	52.5
$A + 5K$ to $A + 6K$	60.5
$A + 6K$ to $A + 7K$	67.5
$A + 7K$ to $A + 8K$	73.5
$A + 8K$ to $A + 9K$	77.5
$A + 9K$ to $A + 10K$	80
$A + 10K$ to $A + 11K$	80
$A + 11K$ to $A + 12K$	79.5
$A + 12K$ to $A + 13K$	78.5
$A + 13K$ to $A + 14K$	77
$A + 14K$ to $A + 15K$	74.5
$A + 15K$ to $A + 16K$	72.5
$A + 16K$ to $A + 17K$	70
$A + 17K$ to $A + 18K$	66.5
$A + 18K$ to $A + 19K$	63.5
$A + 19K$ to $A + 20K$	60
$A + 20K$ to $A + 21K$	56
$A + 21K$ to $A + 22K$	52
$A + 22K$ to $A + 23K$	47.5
$A + 23K$ to $A + 24K$	42
$A + 24K$ to $A + 25K$	38
$A + 25K$ to $A + 26K$	34
$A + 26K$ to $A + 27K$	30
$A + 27K$ to $A + 28K$	26
$A + 28K$ to $A + 29K$	22.5
$A + 29K$ to $A + 30K$	19.5
$A + 30K$ to $A + 31K$	16.5
$A + 31K$ to $A + 32K$	14
$A + 32K$ to $A + 33K$	11.5
$A + 33K$ to $A + 34K$	9.5
$A + 34K$ to $A + 35K$	7.5
$A + 35K$ to $A + 36K$	5.5
$A + 36K$ to $A + 37K$	4
$A + 37K$ to $A + 38K$	2.5
$A + 38K$ to $A + 39K$	2
$A + 39K$ to $A + 40K$	0.5

This table of frequencies is like those hitherto described in this volume, save that two as yet unknown quantities, A and K, appear in the scale for quantity. This difference makes no difference in any formal respect. The table can be treated like any other. Thus its median is in the step "$A + 14K$ to $A + 15K$," at approximately $A + 14.12K$. The mode may be taken as just between the steps, "$A + 9K$ to $A + 10K$" and "$A + 10K$ to $A + 11K$," or at $A + 10K$. The 25 percentile is at $A + 8.8K$. The 75 percentile is at $A + 20.4K$. 75 percentile − mode = $10.4K$. Mode − 25 percentile = $1.2K$. $Q = 5.8K$.

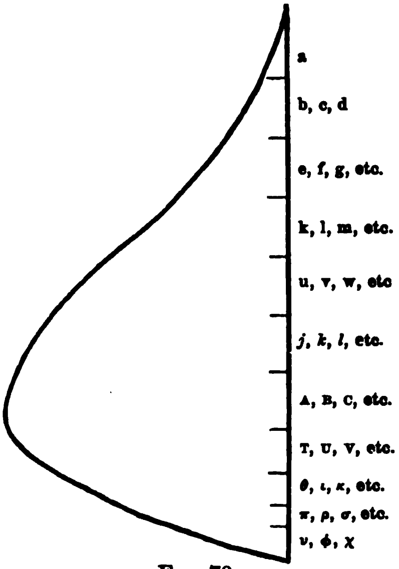


FIG. 70.

The highest-ranking boy, a, will then be represented by the 16.8 of the 1,680 frequency-units at the top, that is, toward $A + 40K$. His ability ranges from $A + 40K$ to $A + 34.7K$ ($16.8 = 0.5 + 2.0 + 2.5 + 4 + 5.5 + 2.3$; and $2.3 = .3$ of 7.5).

The next three—b, c and d—will occupy the next 50.4 of the 1,680 frequency-units, and be included between the limits, $A + 34.7K$ and $A + 30.4K$ ($7.5 - 2.3 = 5.2$; $50.4 = 5.2 + 9.5 + 11.5 + 14.0 + 10.2$; $10.2 = .6$ of 16.5).

Similar calculations can be made for the next six—e, f, g, h, i and j—and so on. The results are shown graphically in Fig. 70.

The average ability of each division may be calculated roughly from the facts obtained in this way. Thus the highest boy, a, being represented by $0.5\ (A + 39.5K)$, $2\ (A + 38.5K)$, $2.5\ (A + 37.5K)$, $4\ (A + 36.5K)$, $5.5\ (A + 35.5K)$ and $2.3\ (A + 34.5K)$,[1] has as an average $A + 36.4K$.[2]

A table can thus be formed as follows:

Boy a has as his approximate ability $A + 36.4K$;
Boys b, c, d have as their approximate ability $A + 32.2K$;[3]
Boys e, f, g, h, i, j have as their approximate ability $A + 28.0K$;
Boys k, l, m, n, o, p, q, r, s, t have as their approximate ability $A + 23.8K$; etc.

So far we have defined or measured the scholarship of each boy as his distance above the low limit (A) of the whole group in terms of K as a unit. All of these measures can be turned into distances *minus* from the upper limit or *plus* and *minus* from the mode, average, median, or any other percentile point on the scale. They can be put in terms of any measure of the variability of the scheme, or of any part of it, instead of K. For instance, the best boy is $+ 26.4K$ or $+ 4.6Q$ from the mode.

The scholarship of every boy in the group can thus be represented in definite quantities of some unit of amount of difference K from some point of reference. This unit itself is definable as "the difference between this given person and that given person." The standard is similarly definable as the scholarship of such and such a given individual.

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[1] A probably more exact value could be assigned to this 2.3, which is the upper three tenths of the rectangle on base $A + 34K$ to $A + 35K$, but the mid-point is accurate enough for our purpose.

[2]
$$\begin{array}{r@{\;+\;}r}
0.5A & 19.75K \\
2.0A & 77.0\ K \\
2.5A & 93.75K \\
4.0A & 146.0\ K \\
5.5A & 195.25K \\
2.3A & 79.35K \\
\hline
16.8A & 611.10K
\end{array}$$

$$\frac{16.8A + 611.1K}{16.8} = A + 36.4K.$$

[3] $[5.2\ (A+34.5K) + 9.5\ (A+33.5K) + 11.5\ (A+32.5K) + 14.0\ (A+31K) + 10.2\ (A+30.5K)]$, when divided by 50.4, gives $A+32.2K$.

By this method the obscurest and most complex traits, such as morality, enthusiasm, eminence, efficiency, courage, legal ability, inventiveness, etc., can be made material for ordinary statistical procedure, the one condition being that the general form of distribution of the trait in question be approximately known.

If now one has a group of individuals ranked by their relative position in the group, his first task before he can transmute the series of relative positions into a series of units of amount is to ascertain the form of distribution. This may be done (1) by measuring enough sample individuals objectively in units of amount, or (2) if the trait can not be measured in units of amount, by inferring the form of distribution from that of similar traits which can be.

1. Suppose one had 2,000 ten-year-old boys measured with respect to intellect by relative position.[4] If now one measured 200 of them objectively with tests scorable in units of amount, he could properly transmute the 2,000 on the basis of the type of distribution found for the 200.

2. Suppose one had 1,000 individuals measured with respect to delicacy of discrimination of sound by relative position. (It is well-nigh impossible to measure sensitiveness to sound in objective units which another observer can duplicate, because of the influence of size of room, resonance, etc.) It is fairly certain from studies of the delicacy of discrimination of length, weight, etc., that delicacy of discrimination of sound is distributed in something approximating sufficiently to a probability surface, with range of from $+ 3\sigma$ to $- 3\sigma$, to prevent calculations on that basis from being more than a little wrong on the average. We may, therefore, transmute the 1,000 measures by relative position into units of amount, on the hypothesis that such is the form of distribution.

The labor of transmutation for cases which follow the probability type of distribution may be rendered almost *nil* by the use of tables. If the probability surface of range $+ 3\sigma$ to $- 3\sigma$ is divided up into 100 equal areas representing the 100 successive per cents. from the highest to the lowest of the total group, and the average distance from the average in terms of σ is calculated for each per cent., the result is Table 21.

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[4] Such measures, at least approximately correct, would in fact be easy to obtain through school marks, teachers' opinions, personal conferences, etc.

TABLE 21

Values, in Terms of the Mean Square Deviation, σ, of each Single Per Cent., the Distribution Being of Form *A*. Beginning with the Extreme

Per cents in Order from Highest Rank or from Lowest Rank toward the Central Tendency	Value in Terms of σ	Per cents in Order from Highest Rank or from Lowest Rank toward the Central Tendency	Value in Terms of σ
1st	2.7	26th	.659
2d	2.18	27th	.628
3d	1.96	28th	.598
4th	1.81	29th	.568
5th	1.695	30th	.539
6th	1.598	31st	.510
7th	1.514	32d	.482
8th	1.439	33d	.454
9th	1.372	34th	.426
10th	1.311	35th	.399
11th	1.250	36th	.372
12th	1.200	37th	.345
13th	1.150	38th	.319
14th	1.103	39th	.293
15th	1.058	40th	.266
16th	1.015	41st	.240
17th	.974	42d	.210
18th	.935	43d	.189
19th	.896	44th	.164
20th	.860	45th	.139
21st	.824	46th	.113
22d	.789	47th	.087
23d	.755	48th	.063
24th	.722	49th	.037
25th	.690	50th	.013

If now we ask, "What will be the average ability of the highest 6 per cent.?" we have only to add the figures for the first 6 per cents. and divide by 6 (the result being 1.99). Similarly to get the average ability of any consecutive series of unit percentages. Table 22 gives the results of such computation for every consecutive series in the upper half of the total group. If the signs are changed to minus it serves for the lower half.

The figures along the top stand each for the percentage already made up in counting in from the extreme. The figures down the side stand for the percentage in the group for which a measure in terms of amount is to be found. The entries in the body of the table stand for the average amount, in terms of σ, of any percentage counted in from any point toward the average. When a percentage passes the average (*e. g.*, 30 per cent. after 40 per cent. have been used up in counting in from the top) it is necessary to take from the table two entries, one for the plus cases down to the average, the other for the minus cases, up to the average, of which the percentage is made up, and from these two entries to compute the average for the given percentage. Thus, 40 per cent. from the upper extreme having been used up, the next 30 per cent. will average.

$$\frac{(+ .13 \times 10) + (- .26 \times 20)}{30}, \text{ or } - .13.$$

Illustrations of the simpler usage in cases not passing the average are as follows:

The first 1 per cent. of a group averages + 2.7
The " 8 " " " " average + 1.86
The 9th and 10th per cents. of a " " + 1.34
Per cents. 6, 7 and 8 from the bottom " − 1.52.

With the aid of Table 22 one can turn measurements by relative position into measurements in units of + or − σ almost as fast as one can read.

For instance, of 800 schoolboys,

8 per cent. received a mark of *E*
20 per cent. received a mark of *VG*
38 per cent. received a mark of *G*
24 per cent. received a mark of *F*
8 per cent. received a mark of *P*
2 per cent. received a mark of *U*

The table tells us at once that, in so far as the distribution of the ability in the group in question follows the form described above (Form *A*),

$$\begin{aligned} E &= +1.86\sigma \\ VG &= +\ .94\sigma \\ G &= +\ .08\sigma \\ F &= -\ .80\sigma \\ P &= -1.59\sigma \\ U &= -2.44\sigma \end{aligned}$$

TABLE 22 (*a*)

	0	1	2	3	4	5	6	7
1	270	218	196	181	170	160	151	144
2	244	207	189	175	165	156	148	141
3	228	198	182	170	160	152	144	137
4	216	191	177	165	156	148	141	134
5	210	185	172	161	152	145	138	131
6	199	179	167	157	149	141	135	129
7	192	174	163	153	145	138	132	126
8	186	170	159	150	142	135	128	124
9	181	165	155	147	139	133	126	129
10	176	161	151	143	136	130	124	111
11	171	158	148	140	134	127	122	116
12	167	154	145	138	131	125	119	114
13	163	151	142	135	128	122	117	112
14	159	147	139	132	126	120	115	110
15	156	144	136	129	123	118	113	108
16	152	141	134	127	121	116	111	106
17	149	139	131	125	119	113	109	104
18	146	136	129	122	117	111	106	102
19	143	133	126	120	114	109	105	100
20	140	131	124	118	112	107	103	98
21	137	128	121	116	110	105	101	96
22	135	126	119	113	108	103	99	95
23	132	124	117	111	106	101	97	92
24	130	121	115	109	104	100	95	91
25	127	119	113	107	102	98	93	89
26	125	117	111	105	101	96	92	88
27	123	115	109	104	99	94	90	86
28	120	113	107	102	97	92	88	84
29	118	111	105	100	95	91	87	83
30	116	109	103	98	93	89	85	81
31	114	107	101	96	92	87	83	79
32	112	105	99	94	90	86	82	78
33	110	103	98	93	88	84	80	76
34	108	101	96	91	86	82	79	75
35	106	99	94	89	85	81	77	73
36	104	97	92	88	83	80	75	72
37	102	96	91	86	82	78	74	70
38	100	94	89	84	80	76	72	69
39	98	92	87	83	79	75	71	67
40	97	91	86	81	77	73	69	66
41	95	89	84	80	75	72	68	64
42	93	87	82	78	74	70	66	63
43	91	85	81	76	72	69	65	62
44	90	84	79	75	71	67	64	
45	88	82	78	73	69	66		
46	86	81	76	72	68			
47	85	79	75	70				
48	83	78	73					
49	81	76						
50	80							

TABLE 22 (*b*)

	8	9	10	11	12	13	14	15
1	137	131	125	120	115	110	106	102
2	134	128	122	118	112	108	104	99
3	131	125	120	115	110	106	102	97
4	128	123	118	113	108	104	100	96
5	126	120	115	111	106	102	98	94
6	123	118	113	108	104	100	96	92
7	121	116	111	106	102	98	94	90
8	118	113	109	104	100	96	92	88
9	116	111	106	102	98	94	90	86
10	114	109	104	100	96	92	88	85
11	111	107	102	98	94	90	87	83
12	109	105	100	96	92	89	85	81
13	107	103	99	94	91	87	83	80
14	105	101	97	93	89	85	81	78
15	103	99	95	91	87	83	80	76
16	101	97	93	89	85	82	78	75
17	99	95	91	87	84	80	77	73
18	98	93	89	86	82	78	75	72
19	96	92	88	84	80	77	73	70
20	94	90	86	82	79	75	72	69
21	92	88	84	81	77	74	70	67
22	90	87	83	79	76	72	69	66
23	89	85	81	78	74	71	67	64
24	87	83	80	76	73	69	66	63
25	85	82	78	74	71	68	64	61
26	84	80	76	73	70	66	63	60
27	82	78	75	71	68	65	62	58
28	80	77	73	70	67	63	60	57
29	79	75	72	68	65	62	59	56
30	77	74	70	67	64	60	57	54
31	76	72	69	65	62	59	56	53
32	74	71	67	64	61	58	54	51
33	73	69	66	63	59	56	53	50
34	71	68	64	61	58	55	52	49
35	70	66	63	60	56	53	50	47
36	68	65	61	58	55	52	49	
37	67	63	60	57	54	51		
38	65	62	59	55	52			
39	64	61	57	54				
40	62	59	56					
41	61	58						
42	60							

TABLE 22 (*c*)

	16	17	18	19	20	21	22	23
1	97	94	90	86	82	79	76	72
2	95	92	88	84	81	77	74	71
3	94	90	86	82	79	76	72	69
4	92	88	84	81	77	74	71	67
5	90	86	82	79	76	72	69	66
6	88	84	81	77	74	71	68	64
7	86	83	79	76	72	69	66	63
8	84	81	77	74	71	68	64	61
9	83	79	76	73	69	66	63	60
10	81	78	74	71	68	65	62	59
11	79	76	73	69	66	63	60	57
12	78	74	71	68	65	62	59	56
13	76	73	70	66	63	60	57	54
14	75	71	68	65	62	59	56	53
15	73	70	66	63	60	57	54	51
16	71	68	65	62	59	56	53	50
17	70	67	64	60	57	54	52	49
18	68	65	62	59	56	53	50	47
19	67	64	61	58	55	52	49	46
20	65	62	59	56	53	50	47	45
21	64	60	58	55	52	49	46	43
22	62	59	56	53	50	48	45	42
23	61	58	55	52	49	46	43	41
24	60	57	54	51	48	45	42	39
25	58	55	52	49	46	43	41	38
26	57	54	51	48	45	42	39	37
27	55	52	49	46	44	41	38	35
28	54	51	48	45	42	39	37	
29	53	50	47	44	41	38		
30	51	48	45	42	40			
31	50	47	44	41				
32	48	46	43					
33	47	44						
34	46							

TABLE 22 (*d*)

	24	25	26	27	28	29	30	31
1	69	66	63	60	57	54	51	48
2	67	64	61	58	55	52	50	47
3	66	63	60	57	54	51	48	45
4	64	61	58	55	52	50	47	44
5	63	60	57	54	51	48	45	43
6	61	58	55	53	50	47	44	41
7	60	57	54	51	48	45	43	40
8	58	55	52	50	47	44	41	39
9	57	54	51	48	46	43	40	37
10	56	53	50	47	44	41	39	36
11	54	51	48	46	43	40	37	35
12	53	50	47	44	41	39	36	33
13	51	48	46	43	40	37	35	32
14	50	47	44	42	39	36	33	31
15	49	46	43	40	37	35	32	29
16	47	44	42	39	36	33	31	28
17	46	43	40	37	35	32	29	27
18	44	42	39	36	33	31	28	26
19	43	40	38	35	32	30	27	24
20	42	39	36	34	31	28	26	
21	40	38	35	32	30	27		
22	39	36	34	31	28			
23	38	35	32	30				
24	36	34	31					
25	35	32						
26	34							

TABLE 22 (*e*)

	32	33	34	35	36	37	38	39
1	45	43	40	37	35	32	29	27
2	44	41	39	36	33	31	28	25
3	43	40	37	35	32	29	27	24
4	41	39	36	33	31	28	25	23
5	40	37	35	32	29	27	24	21
6	39	36	33	31	28	25	23	20
7	37	35	32	29	27	24	21	19
8	36	33	31	28	25	23	20	18
9	35	32	29	27	24	21	19	16
10	33	31	28	25	23	20	18	15
11	32	29	27	24	22	19	16	14
12	31	28	25	23	20	18	15	
13	29	27	24	22	19	16		
14	28	25	23	20	18			
15	27	24	22	19				
16	26	23	20					
17	24	22						
18	23							

TABLE 22 (*f*)

	40	41	42	43	44	45	46	47	48	49
1	24	21	19	16	14	11	09	06	04	01
2	23	20	18	15	13	10	08	05	03	
3	21	19	16	14	11	09	06	05		
4	20	18	15	13	10	08	05			
5	19	16	14	11	09	06				
6	18	15	13	10	08					
7	16	14	11	09						
8	15	13	10							
9	14	11								
10	13									

For any given form of distribution, a table like Table 22 can be constructed, by which any defined position in a series can be transmuted into terms of amount + or − from the mode, in units of the variability.

§ 23. *Transmutation by Means of Knowledge of the Equality of the Steps of Difference*

The Equality of the Least Noticeable Difference.—There is still another possibility of turning measures by relative position into units of amount and so making them available for common scientific usage. In certain cases it may be justifiable to suppose that the least noticeable difference is a constant quantity for any one trait for any one observer; in simpler words, that if I say that John, James and Peter are to me indistinguishable, say, in literary merit, but that Henry and William are a shade better, and that George and Fred are a shade better than Henry and William, the actual difference between *JJP* and *HW* equals that between *HW* and *GF*. In so far as this were true, we could, with a large group of individuals varying continuously from the low to the high extreme, make groups on the basis of the least noticeable difference and call the steps of ability from group to group always the same.

The measures are then identical in form with those by ordinary units of amount. The only difference is that the amount of the quantity at the starting-point of the whole group (*A*) and the amount of the step from one subgroup to the next (*K*) are unknown except from the things measured themselves, and are undefinable except in terms of them. The question, "How much are *A* and *K*?" can be answered only by pointing to the achievements of the lowest group and saying, "That is *A*," by pointing to the differences between that group and others and saying, "This much difference is K, this much $4K$, this much $20K$ and so on," K being the least noticeable difference.

The hypothesis that the least noticeable difference in a trait is for the same observer a constant quantity has not been tested sufficiently to decide how far its use is justifiable, but there can be no doubt that some modification of the principle involved will some time be a valuable resource of the theory of mental measurements.

The Equality of Equally Often Noticed Differences.—Suppose specimens a, b, c, d, e, f, etc., to be ranked in order for trait X a hundred times. The hundred rankings may comprise a hundred judgments by one judge, or one judgment by each of a hundred judges, or ten judgments by each of ten judges, etc., etc., without alteration in the general procedure. Suppose that a is placed below b 74 times, and above b 26 times; suppose that b is placed below c 74 times, and above c 26 times; suppose that c is placed below d 74 times, and above d 26 times. Then if "equally often noticed" can be assumed to mean "equal," the differences, $b - a$, $c - b$, and $d - c$, are equal. Let A = the difference between the amount of trait X possessed by a and the absolute zero for X. Let K = the amount of difference which the observer in question notices 74 times out of a hundred. Then the measure of a is A; that of b is $A + K$; that of c is $A + 2K$, etc. The measures are now identical in form with those by ordinary units of amount. For this method to be applicable, the percentage of observations of a difference must be less than 100, since if two differences are always noticed, one may be very small and one very great. The method as a whole presupposes that the observations are made by judges of some competence. Its precision depends upon how competent they are.

§ 24. *Transmutation by Means of the Amount of Agreement of Different Judges in Respect to the Relative Positions*

Suppose that, in the case quoted in Section 23, the percentages of judgments of difference had been:

a below b,	74
b below c,	74
c below d,	74

d below *e*,	70
e below *f*,	80
f below *g*,	60
g below *h*,	90

Now, for the same reasons which make it allowable, the judges being competent, to infer equality in the differences $b - a$, $c - b$ and $d - c$, we can infer that $e - d$ and $g - f$ are less than $b - a$, $c - b$ and $d - c$; and can infer that $f - e$ and $h - g$ are greater than $b - a$, $c - b$, and $d - c$. We can infer, that is, that $(h - g) > (f - e) > (b - a) = (c - b) = (d - c) > (e - d) > (g - f)$.

It is possible, by making a further assumption, to infer how much greater $g - h$ is than $e - f$, and so on for the other differences in the series. The assumption to be made concerns the relation between the amount of a difference and the percentage of times that it will be noticed. The theories at the basis of any such assumptions are beyond the scope of this book, but the probably best relation to assume is that shown in Table 23. In this table, 1.00 is taken arbitrarily to equal such a difference between two facts, *a* and *b*, that *b* will be judged $> a$ in 75 per cent. of the judgments and $< a$ in 25 per cent. of the judgments. That is, 1.00, or $b - a$, is positive; and is the amount of difference whose direction is noted correctly in 75 per cent. of the judgments. If then the fact *q* is judged greater than *p* in 51 per cent. of the judgments, $q - p$, by the table equals .04; if *w* is judged greater than *v* in 52 per cent. of the judgments, $w - v = .07$; and so on through the table.

TABLE 23

THE AMOUNTS OF DIFFERENCE $(x - y)$ CORRESPONDING TO GIVEN PERCENTAGES OF JUDGMENTS THAT $x > y$. % r = THE PERCENTAGE OF JUDGMENTS THAT $x > y$. Δ/P.E. $= x - y$, IN MULTIPLES OF THE DIFFERENCE SUCH THAT % r IS 75[5]

% r	Δ/P.E.	% r	Δ/P.E.	% r	Δ/P.E.	% r	Δ/P.E.	% r	Δ/P.E.
50	.00	60	.38	70	.78	80	1.25	90	1.90
51	.04	61	.41	71	.82	81	1.30	91	1.99
52	.07	62	.45	72	.86	82	1.36	92	2.08
53	.11	63	.49	73	.91	83	1.41	93	2.19
54	.15	64	.53	74	.95	84	1.47	94	2.31
55	.19	65	.57	75	1.00	85	1.54	95	2.44
56	.22	66	.61	76	1.05	86	1.60	96	2.60
57	.26	67	.65	77	1.10	87	1.67	97	2.79
58	.30	68	.69	78	1.14	88	1.74	98	3.05
59	.34	69	.74	79	1.20	89	1.82	99	3.45

[5] A more elaborate table for this same purpose is given in Appendix II.

In our illustrative case, we have, from Table 23:

$$\begin{aligned} b - a &= .95 \\ c - b &= .95 \\ d - e &= .95 \\ e - d &= .78 \\ f - e &= 1.25 \\ g - f &= .38 \\ h - g &= 1.90 \end{aligned}$$

and, letting A equal the difference between a and the absolute zero for the trait in question,

$$\begin{aligned} a &= A \\ b &= A + .95 \\ c &= A + 1.90 \\ d &= A + 2.85 \\ e &= A + 3.63 \\ f &= A + 4.88 \\ g &= A + 5.26 \\ h &= A + 7.16 \end{aligned}$$

wherein 1.00 equals a difference a trifle greater than $b - a$, $c - b$, or $d - e$, or such a difference as 75 per cent. of the judgments in question would note correctly.

For reasons not to be stated here it is best to avoid relying on this table outside the limits of 65 and 85 for the percentages of judgments that one fact is greater or less than another (per cent. r).

Problems

29. Using Table 21, calculate the measure in terms of units of amount: (1) of the highest four per cent. of a group normally distributed; (2) of the six per cent. just above the mode; (3) of the three per cent. from the end of the seventeenth down—that is, of the 18th, 19th and 20th per cents. Verify the results from the entries for these groups in Table 22.

30. Construct the beginning of a table like Table 22 for the form of distribution (Form D) shown in Fig. 23 and Table 11, putting in the entries for percentages 1, 2, 3, 4 and 5, for each of the three cases of 0, 1 %, and 2 % already used. Begin at the extreme of greater skewness. Be accurate to the first decimal (tenths of σ).

31. Suppose 100 individuals to be ranked in order as follows:

a is the best
b, *c*, *d*, *e* are the next, and are indistinguishable in merit.
f, *g*, *h*, *j* are the next, and are indistinguishable in merit.
k, *l*, *m*, *n*, *o*
p, *q*, *r*, *s*, *t* } are the next, and are indistinguishable in merit.
etc, etc.

Supply the approximate values in the following:

a is + _*Q* from the median if the distribution is a rectangle.
b is + _*Q* from the median if the distribution is a rectangle.
f is + _*Q* from the median if the distribution is a rectangle.
s is + _*Q* from the median if the distribution is a rectangle.

a is + _σ from the median or Av. if the distribution is of Form *A*.
b is + _σ from the median or Av. if the distribution is of Form *A*.
f is + _σ from the median or Av. if the distribution is of Form *A*.
s is + _σ from the median or Av. if the distribution is of Form *A*.

a is + _σ from the mode if the distribution is of Form *D*.
b is + _σ from the mode if the distribution is of Form *D*.

32. On the hypothesis that the distribution of darkness of eyes is of Form *A*, use Table 22 and transmute into terms of units of amount the following measures by relative position:

Eye Color	Per Cents. of Englishman[6]
Light blue	2.9 call 3
Blue. Dark blue	29.3 call 29
Gray. Blue green	30.2 call 30
Dark gray. Hazel	12.3 call 12
Light brown. Brown	11.0 call 11
Dark brown	10.8 call 11
Very dark brown. Black	3.6 call 4

It is possible, by interpolating, to use the table for a finer scale than to a single per cent. But it is hardly worth while.

33. Suppose a group of individuals to receive grades as follows in a trait in which variation is continuous: 2 per cent. received *A*; 22 per cent. received *B*; 44 per cent. received *C*; 25 per cent. received *D*; 6 per cent. received *E*; 1 per cent. received *F*. Suppose that this grouping is by equally often noticed differences and that the differences can be assumed equal. Calculate the approximate values to complete the following:

a grade of *A* = _*Q* from the Median.
a grade of *B* = _*Q* from the Median.
a grade of *C* = _*Q* from the Median.
a grade of *D* = _*Q* from the Median.
a grade of *E* = _*Q* from the Median.
a grade of *F* = _*Q* from the Median.

34. Suppose that a certain man, *Z*, whose life was fully known, was, by the average opinion of a hundred statesmen, scientists and men of general wisdom, rated as having just barely contributed a

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[6] From Galton's "Natural Inheritance."

balance of one trivial satisfaction to a balance of one person in the world, past, present and future. Suppose that by the same hundred expert judges of the services performed by human individuals, we have the lives of twenty-five men, Y, X, W, V, etc., up to A, A being Pasteur, rated as having performed services such that $A - B$, $B - C$, $C - D$, $D - E$, etc., are all equal.

Suppose that you knew the lives of two men—*a* and *b*. How would you answer the question: *How many times as great was a's service to the world as b's?* Under what conditions would your answer be true? What factors would work to make it depart from the truth?

CHAPTER IX

The Measurement of Differences and of Changes

The chief questions that concern the measurement of differences in the mental sciences arise in the case of (1) comparisons of groups in respect to the amount of some trait which they display, (2) the comparison of individuals or groups in respect to variability, and (3) measurements of changes. Instead of any general abstract treatment of the measurement of differences, therefore, I shall present the special applications of it to these three problems. Only a very brief outline of the problem as a whole will be given as an introduction.

§ 25. *The Varieties of Differences to be Measured*

The difference between any two amounts of the same kind of fact may be measured. The amounts may be:

1. Two single figures, each standing for a central tendency, *e. g.*, averages, medians or modes.
2. Two single figures, each standing for a variability, *e. g.*, A.D.'s, σ's or P.E.'s.
3. Two single figures, each standing for a difference itself.
4. Two single figures, each standing for a relationship.
5. Two total distributions, each standing for a general tendency plus the deviations from it.

The central tendency may be to the possession of a certain amount of variability, of difference or of relationship, as well as of a thing or quality. It will, however, commonly be the latter.

The classification above could, of course, be extended *ad infinitum* with such complexities as: "The measurement of the difference between two variabilities, each being of the amounts of relationship between the amount of difference between (1) 10-year-olds and 11-year-olds in motor ability and (2) 10- and 11-year-olds in sensory discrimination."

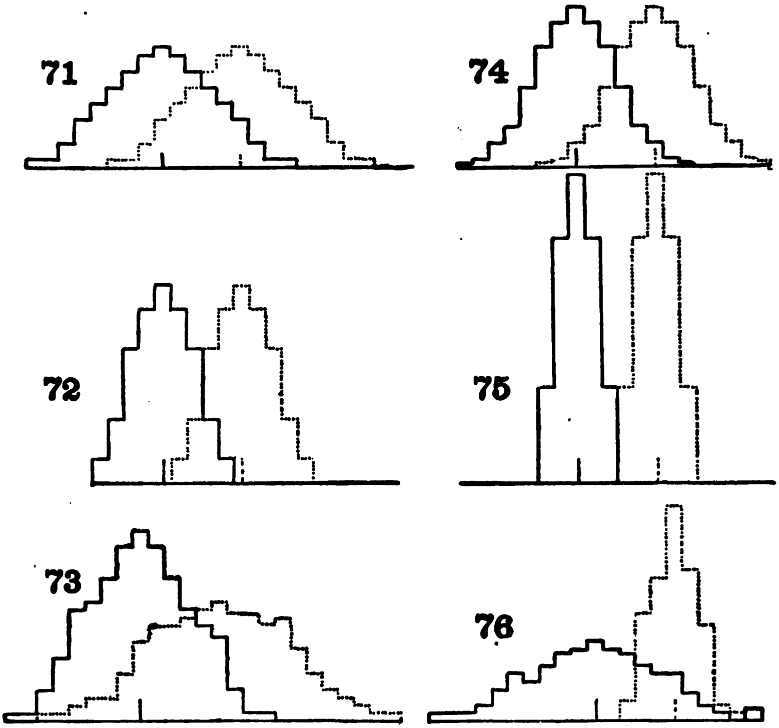
The difference between two single figures will be measured (*a*) by the gross difference, or (*b*) by the percentage which the gross difference is of the amount of one of them, or (*c*) by the percentage which it is of some other feature of one of them. The difference between two total distributions will be measured fully by comparing them item by item; the difference may be summarized in various ways.

The difference between two facts, each of which is measured by its relative position in a series, may be measured most satisfactorily by transmuting the series into measures in units of amount and then using regular methods.

§ 26. *The Comparison of Groups*

The Importance of Measuring the Amount of Overlapping.— The common custom of comparing groups by stating only their central tendencies is inadequate because, for both practical and theoretical purposes, the meaning of a difference between the two central tendencies depends upon the variabilities of the groups. The mere fact, for example, that, in a test in cancelling the *A*'s on a page of mixed capital letters, the averages for 12-year-old boys and for 12-year-old girls respectively were 41 and 46, might mean (1) that the lowest ranking girl was above the highest boy— *i. e.*, that boys and girls were in this trait totally distinct species— or (2) that only 5 per cent. of girls were better than the highest ranking boy, or even (3) that no girl was equal to the highest ranking boy. It might mean,in fact, all sorts of conditions, some of which are pictured in Figs. 71 to 76.

It is of no great advantage to estimate the difference as a percentage rather than a gross amount. One group may, in ten different tests, have always an average twenty per cent. higher than the other, and yet the differences in ability may really be equal in no two of the ten cases. Since, in mental and social traits, there are rarely absolute zero points at which to start the scale, the meaning of each percentage will depend upon the number chosen as the starting-point in measuring. We can always make a difference so expressed seem less by starting the scale at 10 or 40 or 100 instead of at 0 or 4 or 10. For instance, if the *A* test is scored by the number of *A*'s marked, the percentage superiority of girls to boys is 12.2; if by the number marked more than the lowest 12-year-old record, it is 18.5; if by the number of *A*'s omitted, it is 8.5. Clearly the figure depends on an entirely arbitrary factor. Also, a given percentage in a case where the variability of the trait is great will always mean for practical purposes a less difference than it does in a case where the variability is small.



FIGS. 71–76. Graphic comparisons of six pairs of groups, the difference between the averages being in all cases the same.

In addition to the difference between the two central tendencies, we need some measure which will inform us of the extent to which the two groups overlap—the extent, therefore, to which treatment applicable to one group will or will not be applicable to the other.

Such a measure is got by comparing the two total distributions or, approximately, in the case of traits similar in their form of distribution, by stating the variabilities of the two groups as well as their central tendencies. Thus, to use our previous illustration, the distribution of 12-year-old boys and of 12-year-old girls in the *A* test as given in Table 24 and Fig. 77, tells us at once that the difference between the averages is 5.2, that over 99 per cent. of the girls are contained between the same limits of ability as the boys, that only 31 per cent. of boys reach the median mark for girls, that the sex difference is far less important practically than individual differences within either sex, that between 28 and 62 are 89 per cent. of the boys and 87 per cent. of the girls. These same measures could be obtained approximately from the theoretical properties of the surface of frequency of Form *A*, if the variabilities of the groups were given instead of the total distributions.

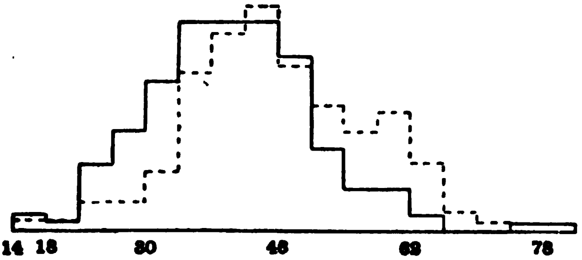


FIG. 77. The continuous line gives the distribution of ability in perception (*A* test) in 12-year-old boys; the dotted line that for girls. The cases are grouped more coarsely than in Table 24.

The Importance of Obtaining Commensurate Measures of Difference.—The comparison of groups is often made in order to answer such questions as: Does group *A* differ from group *B* more in trait I. than in trait II.? How much more? Does group *A* differ from group *B* in trait I. more than group *C* differs from group *D* in trait I.? How much more? There is also the very important special case where groups *A* and *B* are compared in the same general trait, I., but with different tests, I*a*, I*b*, I*c*, used as symptoms of it. The measure of the difference between *A* and *B* should be, so far as is possible, commensurate with any measures of the differences between *A* and *B* in other traits, and with any measure of the difference between *C* and *D* in the same trait. The best approximation to such commensurability is secured by measuring the difference in terms of the percentage of one group reaching or exceeding the median mark for the other group (or some other set measure). If in Latin, Greek, algebra and history one group of students always show 30 per cent. reaching the median of another group, then it is fair to say that the second group is equally superior in all four of these studies. At least there can be no better evidence than this of equality in amount of difference in mental traits.

TABLE 24

A's Marked in 60 Seconds

	Frequency	
Quantity	For 12-year-old Boys	For 12-year-old Girls
14 up to 16		1
16 up to 18	2	
18 up to 20	1	1
20 up to 22		
22 up to 24	4	2
24 up to 26	4	1
26 up to 28	3	2
28 up to 30	9	1
30 up to 32	10	2
32 up to 34	8	4
34 up to 36	10	11
36 up to 38	15	5
38 up to 40	15	9
40 up to 42	10	11
42 up to 44	13	9
44 up to 46	12	14
46 up to 48	13	10
48 up to 50	8	7
50 up to 52	4	6
52 up to 54	6	7
54 up to 56	3	6
56 up to 58	2	4
58 up to 60	1	8
60 up to 62	4	4
62 up to 64	1	4
64 up to 66	1	3
66 up to 68		1
68 up to 70		1
70 up to 72		
72 up to 74		1
74 up to 76	1	
76 up to 78		
78 up to 80	1	

Under the present conditions of thoughtless measurements of mental traits it frequently happens that groups will be compared with respect to the same trait by different tests, and no one will be able to tell how far results agree. If the mere averages were replaced by the measure *per cent. of group A reaching median of group*

B, results by all sorts of methods could be put together. It is, of course, true that when one group so far exceeds another that its lowest score is above the highest score of the other, the method suggested here fails. Such cases are, however, extremely rare in the comparisons of groups characterized by differences of sex, training, age, social conditions, birth, occupation, locality, etc., such as psychology, education and sociology are studying. In the rare cases of no overlapping of the two distributions, the results from different tests may be made commensurate, so far as is possible, by expressing the differences in terms of the variability of one of the two groups.

Comparison by the percentage of one group that reaches or exceeds the median measure of some other group has the further advantage of being applicable to groups measured by relative position only. For instance, if one knew that the crimes in one town were as listed below in column 1, and those of a second town as listed in column 2, he could state that almost 59 per cent. of the first town's crimes were greater than the median crime of the second, could thus have a quantitative comparison of the two without having to adopt speculative equivalents of one crime in terms of others.

Offense	1 Frequency in First Town	2 Frequency in Second Town
Peddling without a license	2	3
Failure in jury duty	4	5
Disturbing the peace	9	11
Drunkenness	23	28
Robbery	30	27
Assault and robbery	17	11
Arson	8	10
Murder in second degree	5	4
Murder in first degree	1	1
Patricide	1	

§ 27. *Differences in Variability*

In comparing individuals or groups with respect to variability, allowance may have to be made for the fact that the amount of the central tendency influences the size of the σ or A.D. or P.E. or Q that is obtained. For instance, 22 individuals added for 40 seconds, and gave a group-score of—*Median, 9.0; A.D., 2.18.* The same 22 individuals then added for 80 seconds and gave a group score of—*Median, 16.0; A.D. 3.41.* In a final test for 120 seconds, the results were—*Median, 23.5; A.D., 5.18.* These figures do not mean that the real variability of the group doubled within a few minutes, or that it altered at all, but only that the gross amount of the variability depends upon the gross amount of the measures themselves as well as upon the real variability. The gross amount of variability in the length of the line drawn by a group of individuals trying to equal a 100-mm. line will be far less than the gross variation of their attempts to equal a 1,000-mm. line, yet the real variability is presumably about the same.

Karl Pearson has proposed, as a measure of variability by which individuals or groups may be fairly compared, the gross variability divided by the average. By this figure, which we may call the Pearson Coefficient of Variability, we should, in the case of the 12-year-old boys and girls in the *A* test (Boys, Av. 40.7, A.D., 8.1; Girls, Av. 45.9, A.D., 8.5) reverse the gross difference, the girls becoming only 93 per cent. as variable as the boys. It seems to the author more in accord with both theory and facts to use the gross variability divided by the square root of the average.[7]

Further, it can be shown that no one coefficient of variability suffices for all comparisons. In some cases the factors which make the central tendency larger seem to work to make the variability actually *smaller*. Thus, if, from the same race living under the same conditions a group of tall men and a group of short men are picked (at random so far as variability is concerned) by picking men with very long fingers and men with very short fingers, the tall men show a gross variability that is *less* than that of the short men. On the other hand, men of long arm-span show a gross variability in arm-span greater than that of men of short arm-span to such an extent as to require the full allowance of the Pearson coefficient of variability. Correct allowance for the magnitude of central tendencies when comparing their variabilities has, then, to be a product of special consideration of the particular facts in hand. In the case of mental and social measurements whose absolute zero points are undetermined, the allowance is particularly liable to error.

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[7] Samples of such facts will be found in the author's "Empirical Studies in the Theory of Measurement," § 4.

Comparisons of groups in variability are of two sorts: (1) Of different groups with respect to their variabilities in the same trait. (2) Of the same group with respect to its variabilities in different traits.

In the first case the differences between the averages in the cases which interest the student are commonly not very great, and the zero points, though arbitrary, are subject to not very great fluctuations; consequently the comparison by any method is commonly such as to reveal any marked difference in variability that exists. In practise one can do no more than present the two total distributions the variabilities of which are to be compared, explain what zero points were taken and why, and calculate for the reader the relation of the group's variabilities by all three methods.

The second case will only rarely be an important object of study. This is fortunate, since here the differences between central tendencies may run to any amount, and the zero points for some of the traits may be subject to extreme variations. For instance, suppose that one wished to compare the variabilities of adult men in salary and intellect—that is, to answer the questions: "Do men vary more in the amount of salary received than in the amount of intellect possessed? If so, how much more?" In practise one can do no more with such cases than to present the total distributions, explain what zero points were taken and why, and use proper logic in inferring anything about the relations of the variabilities found.

§ 28. *The Measurement of Changes*

By a change in anything is meant the difference between two conditions of it. It might seem that the problem of the measurement of changes was identical with that of measuring differences, and that this section was superfluous. In a certain sense this is true. The general principles of previous sections do answer the special questions of this section. But it will be clearer, and in the end save the student's time, to study these special questions separately, especially since in studies of change one is commonly concerned with a number of successive steps of difference, and is trying to measure, not a single alteration, but a continuous process of alteration.

The Measurement of a Change in an Individual.—A mere series of central tendencies does not give the data for a complete measurement of the change. The averages might be the same and yet the constancy of performance of the individual might have altered. Thus the average values of a stock from 1890 to 1900 might be alike and yet it might have changed from a fluctuating uncertainty in 1890, with, say, an average deviation of 40, to a steady assured value in 1900, with an average deviation of only 3. The stock in 1890 would be more desirable property than the stock in 1900 from the point of view of one moved by the gambler's instinct; the reverse would hold for a steady-going man with a family or for a conservative bank. To measure change fully one needs a series of total distributions. If they are not at hand one must be sure not to pretend to measure something other than that represented by the series of quantities he does have.

Inequalities in units are more likely to escape attention in measurements of change than anywhere else. Yet it is just in such measurements that they may do the most harm. For instance, all statistics with which I am acquainted measure the change in the death-rates from various diseases by series of figures, each giving the proportion of deaths to cases, or to total population, or to some other standard, as in the following:[8]

In 1891, 22.5 per cent. of those having diphtheria died.
In 1892, 22.2 per cent. of those having diphtheria died.
In 1893, 23.3 per cent. of those having diphtheria died.
In 1894, 23.6 per cent. of those having diphtheria died.
In 1895, 20.4 per cent. of those having diphtheria died.
In 1896, 19.3 per cent. of those having diphtheria died.
In 1897, 17.0 per cent. of those having diphtheria died.
In 1898, 14.8 per cent. of those having diphtheria died.
In 1899, 14.2 per cent. of those having diphtheria died.
In 1900, 12.8 per cent. of those having diphtheria died.

Such figures can not be taken at their face value; for to cure one case of diphtheria is not the same quantity of progress as to cure another. The progress of medicine and hygiene which reduces the death-rate from 40 to 30 does so presumably often by curing the easiest quarter of those previously uncured. The next cases will be harder, and possibly to cure the last one of the forty would mean more advance in medicine and hygiene that was needed for the

[8] "London Statistics," Vol. XII., p. 97 of the Medical Officer's Report.]

curing of all the other thirty-nine. When the change is in number of individuals affected or number of errors made or number of tasks done, there is then special danger in neglecting the inequalities among the units; for the change will commonly single out the easiest first.

The common absence of zero points in the case of mental measurements often makes it unwise to express changes in percentile increments, and definitely unjustifiable so to express them if the gross amounts whence the percentages are derived are not also given. If, for instance, I am informed that *A*'s reaction time improved 10 per cent. per year over a given period, I am at a loss to tell what is meant.

In comparing two (or more) individuals with respect to change one may use gross change, percentile change or change in terms of the variabilities of the individuals, provided that he makes it clear which he is using and, of course, treats both individuals alike. No one method is the correct one; all are correct, but measure different things. 4 to 5 equals 8 to 9 if by change is meant amount added; 4 to 5 equals 8 to 10 if one means proportion added; 4 to 5 (the A.D. of 4 being 2) equals 8 to 9.5 (the A.D. of 8 being 3) if one means distance traversed toward the extreme ability of the previous condition. This is all that can be said in general. Each special case may offer reasons for preferring one method. The beginner in statistical work may well use all three.

The Measurement of a Change in a Group.—This heading is ambiguous in that it may be taken to refer: (1) to the measurement of the changes undergone by a series of individuals, or (2) to the change undergone by some measure of a group. It should be needless to say that the two questions are radically different, but they are often confused. The changes in stature of 100 boys from the age 15 to the age 16 are not the change from the average stature of the group 100 boys at 15 to the average stature of the same group at 16 years. The first fact, the total fact of all the individual changes, is calculated from 100 individual measures of *change*, is a distribution with an ascertainable variability, and in all respects stands in the same relation to individual changes as does the distribution of an ability in a group to the abilities of its members. The second fact is calculated as the difference of two averages, has no known variability, is, in fact, simply a partial measure of difference between two groups. If our argument is ever to return to individual changes, the first sort of measure must be used. This will commonly be the case.

For an example take the case of the change in stature of 25

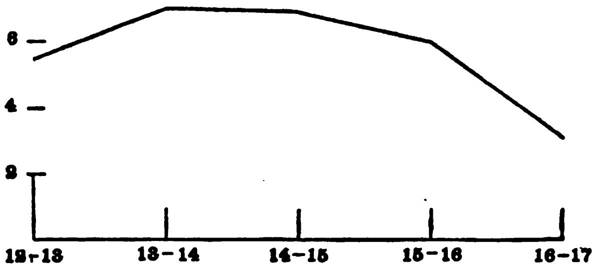

FIG. 78. The heights of the line above the base line at the points 12–13, 13–14, 14–15, 15–16, 16–17, give the differences between the average height at 12 and that at 13, the difference between the average height at 13 and that at 14, etc., for 25 boys measured annually for five years.

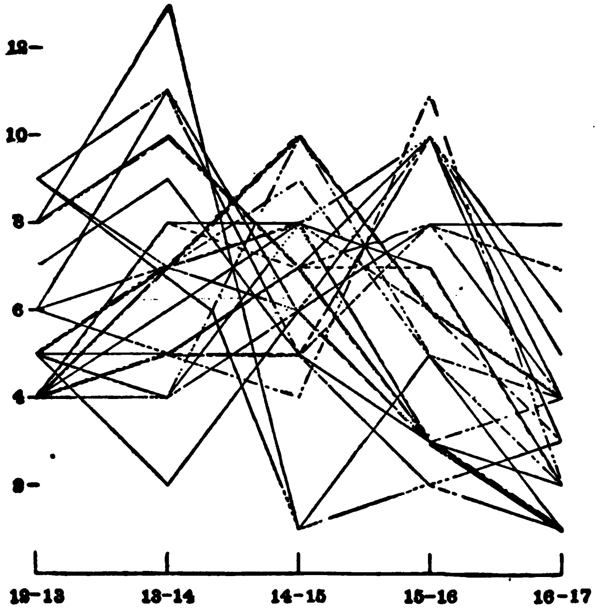

FIG. 79. The heights of the five points, above 12–13, etc., of each line measure the yearly differences for one individual as did the line of Fig. 78 the yearly differences for the average stature of the group. The figure, that is, presents graphically the facts of Table 25.

boys from the twelfth to the seventeenth year.[9] If we try to infer anything about growth from the change in average stature, we have only the following facts: Average stature for 12, 13, 14, 15, 16 and 17 year old boys, 142.6, 148.12, 154.92, 161.60, 167.64 and 170.76 centimeters respectively. Yearly differences, + 5.52, + 6.8, + 6.68, + 6.04 and + 3.12 centimeters. These differences are shown in Fig. 78.

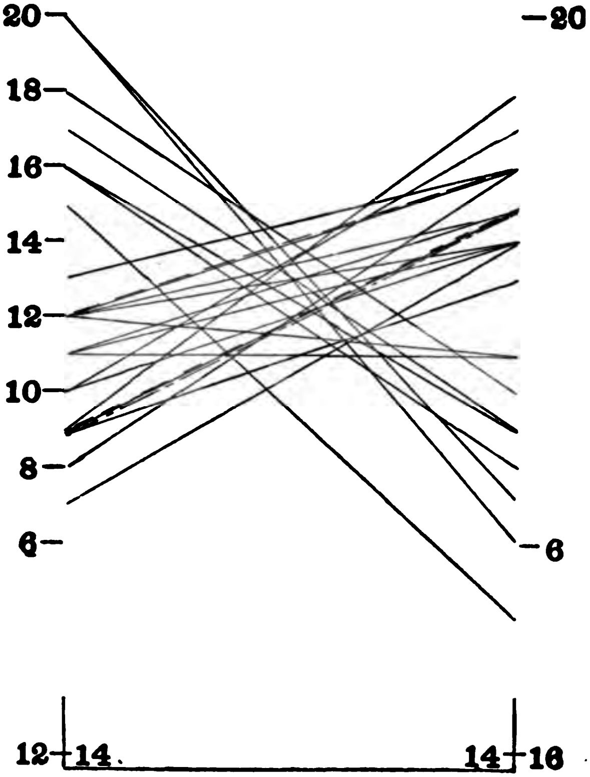

FIG. 80. The height of any one of the lines at its left-hand extreme measures the change in stature of one boy from 12 to 14; its hand at the right-hand extreme measures the change from 14 to 16

If, on the other hand, we preserve the individual changes in our statement, we have the facts of Table 25. These show the great variability in growth and the law of compensation that "boys who were tall at 12 years grow the faster during the interval 12 to 13 and 13 to 14; but during the intervals of 14 to 15 and 15 to 16 they grow slowly; with the boys of short stature at 12 the rates of growth are exactly the reverse."[10] How the single yearly differences above fail to represent the real complexity and correlation of the facts can be seen by comparing Fig. 78 with Fig. 79, which shows the real changes of the 25 individuals. Fig. 80 brings out more clearly the inverse relation between the change from 12 to 14 and that from 14 to 16.

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[9] For these measurements I am indebted to the kindness of Professor Franz Boas and Dr. Clark Wissler.

TABLE 25

GROWTH OF 25 BOYS FROM THE 12TH THROUGH THE 17TH YEAR

BOY	STATURE AT			CHANGE		
	12	12-13	13-14	14-15	15-16	16-17
a	132	5	7	10	6	4
b	134	5	5	7	10	3
c	135	5	2	6	8	8
d	135	6	7	10	6	4
e	136	4	8	8	7	2
f	136	7	9	5	3	2
g	137	4	6	8	6	4
h	137	5	4	8	10	5
i	139	4	8	7	7	2
j	140	5	7	10	6	3
k	142	9	7	6	3	1
l	142	4	5	5	10	6
m	143	4	5	5	8	7
n	144	6	11	6	3	1
o	145	6	5	7	8	4
p	146	4	4	6	10	4
q	146	4	7	8	3	1
r	146	4	5	4	11	2
s	146	9	11	5	2	1
t	147	4	7	9	5	4
u	147	8	10	7	3	1
v	149	7	13	1	5	1
w	151	5	7	8	3	4
x	152	5	4	10	5	2
y	158	9	6	1	3	2

For the measurement of change in a group (that is, of all the individual changes), the statistical treatment is, as suggested above, simply that for any variable fact, the fact here being an amount of change instead of an amount of a thing or condition.

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[10] "The Growth of Boys," by Clark Wissler, *American Anthropologist* (New Series), Vol. 5, pp. 83 and 84.

For the measurement of change from one condition of a group to another the statistical treatment is simply that described in the case of the measurement of differences.

PROBLEMS

35. If we take (1) men (criminals) who are all 74 inches tall and measure the finger-length of each of them, they will vary around their central tendency for finger-length. If we take (2) men (of the same general group, criminals) who are all 62 inches tall and measure the finger-length of each of them, they will vary about their central tendency for finger-length. Their central tendency will be to a much shorter finger length than that of group (1).

What do you infer from the following data, giving the gross varibilities of certain groups obtained in this way?

A group with a C.T. of 107 in finger length had a variability of 35.7
A group with a C.T. of 108 in finger length had a variability of 40.4
A group with a C.T. of 109 in finger length had a variability of 30.9
A group with a C.T. of 111 in finger length had a variability of 32.5
A group with a C.T. of 112 in finger length had a variability of 34.7

. . . .

. . . .

. . . .

A group with a C.T. of 119 in finger length had a variability of 34.5
A group with a C.T. of 120 in finger length had a variability of 33.4
A group with a C.T. of 122 in finger length had a variability of 32.2
A group with a C.T. of 124 in finger length had a variability of 33.3
A group with a C.T. of 126 in finger length had a variability of 31.8

CHAPTER X

THE MEASUREMENT OF RELATIONS

§ 29. *Case I. The Relation of B to A, B and A Being Referable to Absolute Zero Points and the Amounts of B Corresponding to a Given Value of A Being Closely Similar*

The following case may serve as an illustration:

$n =$ the index of refraction of air.
$d =$ the density of air.
p (a quantity subject to the control of the experimenter) $= C_1 d$.
N (a quantity measurable by the experimenter) $= C_2(n - 1)$.
C_1 and C_2 are constants.

The experiments consisted in varying p and measuring the related changes in N. The results are as follows:

When p is 9.989 N is 316.7
When p is 10.146 N is 321.2
When p is 10.163 N is 321.6
When p is 18.281 N is 579.2
When p is 18.365 N is 582.7
When p is 26.932 N is 852.6
When p is 35.990 N is 1142.1
When p is 48.780 N is 1545.1

If each of these pairs of related values is turned into an equation of the form $N = xp$, the results are:

$N = 31.70p$
$N = 31.66p$
$N = 31.64p$
$N = 31.68p$
$N = 31.72p$
$N = 31.66p$
$N = 31.69p$
$N = 31.68p$

Obviously, a single equation $N = 31.68p$ expresses very closely the relationships found for different values of p.

The measurements of relationship here are, of course, not absolutely free from variability. For instance, the $N = 31.70p$ came really from several measurements with an appreciable dispersion. But the dispersion was very small and presumably due entirely to variations in the instruments or process of observation.

If the pairs of values are plotted as in Fig. 81, the slope of the line shows the relationship. The equation $N = 31.68p$ expresses very closely the slope of this line referred to its coordinates. N/p is thus constant. $(n - 1)/d$ equals N/p times some constant. Therefore, $(n - 1)/d$ itself equals a constant. The relation between the index of refraction of air and its density is then such that $(n - 1)/d = k$ or $n = kd + 1$.

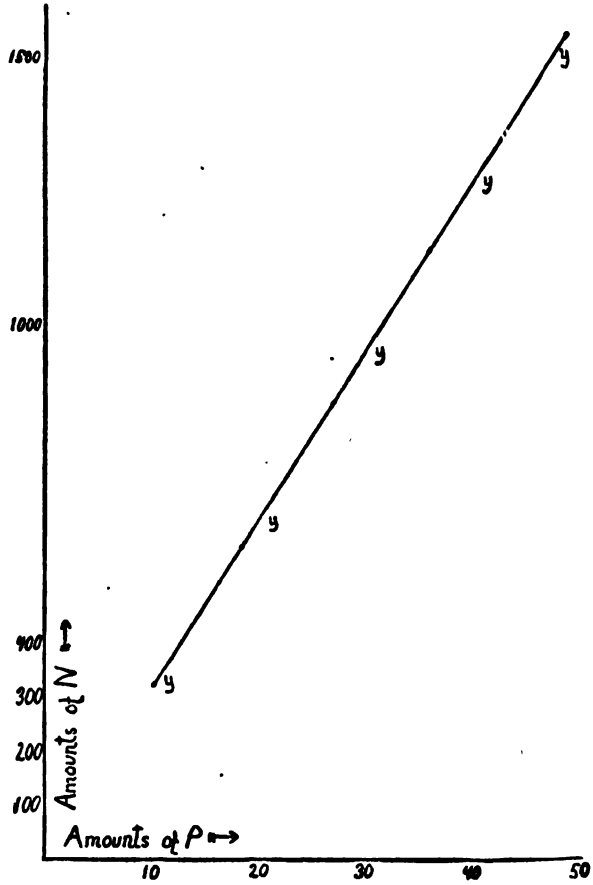

FIG. 81.

§ 30. *Case II. The Relation of B to A, B and A Being Referable to Absolute Zero Points, but the Amounts of B Corresponding to a Given Value of A Being Widely Dispersed*

Consider Table 26 and Fig. 82, which give the percentages of time saved in relearning certain lists of nonsense syllables after various intervals, according to the experiments of Ebbinghaus. The case is identical in form with Case I., save that the variations in the percentage of time saved corresponding to any one interval comprise a wide range of values in the different tests. Instead of getting in different tests, almost exactly the same saving of time in relearning after .32 hr., Ebbinghaus got from 44.7 per cent. to 64.4 per cent. Around the average saving of 58.2 per cent. there was a very wide dispersion, much more than could have been due to the watch used or to the process of observation of the time of beginning and ending. Similarly for the dispersion around 44.2 per cent. (the average percentage saved after 1.05 hrs.), and so on.

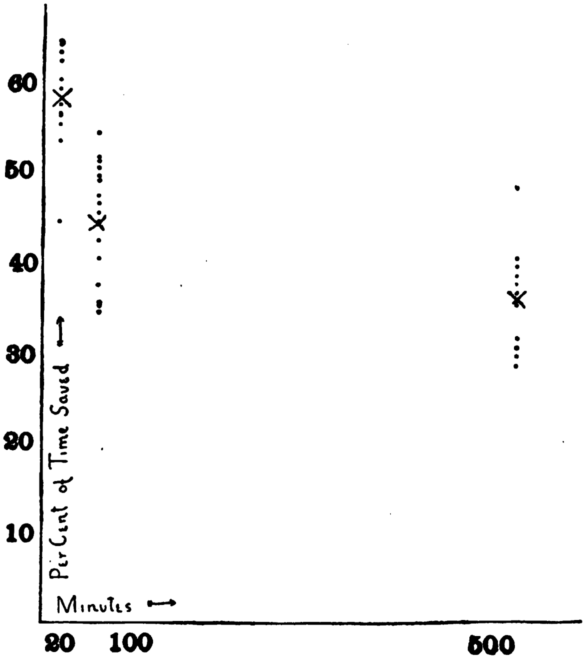

Fig. 82.

In such cases, it is customary to replace the list of values of B corresponding to any given value of A by their central tendency. This procedure should be accompanied by an adequate account of the dispersion around each of these central tendencies.

TABLE 26

RELATION BETWEEN LAPSE OF TIME AND MEMORY[1]

The entries in the table give each the percentage of time saved (from the original time for learning) when a series was re-learned, after the interval, under which the entry stands.

0.32 hrs.	1.05 hrs.	8.75 hrs.	24 hrs.	48 hrs.	144 hrs.	744 hrs.	
64.3	49.6	36.0	26.4	17.4	21.0	26.0	20.0
55.9	37.4	29.0	39.6	32.7	31.1	31.6	19.4
56.6	47.4	28.0	35.4	12.3	32.7	34.7	22.9
62.5	46.8	30.4	39.9	28.9	24.4	31.6	6.7
60.7	51.4	39.8	34.9	30.6	17.7	30.3	6.9
63.1	49.1	35.6	38.9	46.0	5.9	20.5	25.9
59.1	44.5	48.2	46.7	23.5	34.1	10.1	18.9
56.0	54.5	31.6	16.7	25.4	33.3	6.8	20.5
64.4	42.3	35.5	21.3	18.4	28.7	6.5	11.4
44.7	40.9	40.1	38.6	23.4	23.2	13.3	17.3
53.6	34.2	37.9	29.0	41.0	40.3	17.7	17.1
57.7	45.4	38.0	37.8	29.5	37.9	17.1	32.8
	35.8		36.5	33.9	26.5	15.9	31.4
	35.9		29.7	44.9	20.1	27.6	16.4
	51.3		37.0	17.5	39.7	13.2	36.2
	50.0		14.9	42.4	2.5	27.6	13.4
			45.6	6.4	36.2	23.6	31.0
			30.1	22.8	5.3	20.9	7.9
			24.6	31.6	27.9	24.8	36.9
			37.0	30.2	19.0	25.0	14.1
			44.4	19.7	21.0	25.2	6.7
			45.8	31.9	31.4	43.7	16.7
			30.6	14.8	19.7	23.7	
			42.5	32.3	20.9		
			19.8	37.6	24.4		
			32.1	26.7	34.8		
Averages,							
58.2	44.2	35.8	33.7	27.8	25.4	21.1	

§ 31. *Case III. The Relation of B to A, When Neither is Referable to an Absolute Zero Point, but When Amounts of B Corresponding to Any Given Value of A Are Closely Similar*

Suppose it to be true that, in two respects (A and B), scored from w and z as arbitrary zero points, the score a person obtained in B was always 3/10 of the score which he obtained in A. That is, calling x the score in A reckoned from the absolute zero of A, and calling y the score in B, reckoned from the absolute zero of B,

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[1] From Herm. Ebbinghaus, "Über das Gedächtniss," pp. 93–103.

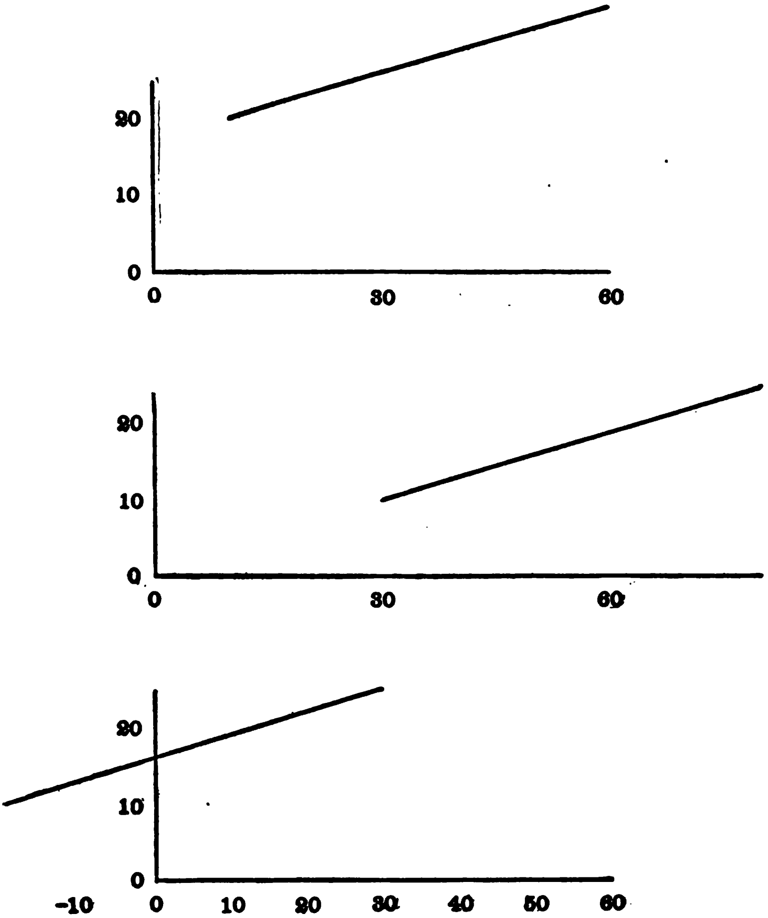

FIG. 83. The relation of B to A scored from arbitrary zero points, w and z, as it would be if referred to absolute zero points. In all three diagrams the vertical line is the scale for B; the horizontal line is the scale for A; their place of meeting is the absolute zero for A and for B. The slant line represents the relation—in the top diagram for $w = 10$ and $z = 20$; in the middle diagram, for $w = 30$ and $z = 10$; in the bottom diagram, for $w = -20$ and $z = 10$.

we have $y - z = .3\ (x - w)$, so that in a series of related pairs we would find such a relation as:

Related Pair	Score in A Measured from w	Score in B Measured from z
a	20	6
b	22	6.6
c	30	9
d	45	13.5, etc.

Then the relation of *B* to *A*, for values of *B* from 0 to 50, can be expressed as any one of the diagrams of Fig. 83, according to the values of the unknown quantities, *w* and *z*, which represent the distances of the arbitrary zeros of the *A* scale and the *B* scale from their absolute zeros.

§ 32. *Case IV. The Relation of B to A, When Neither is Referable to an Absolute Zero Point and When the Amounts of B Corresponding to Any Given Value of A Are Widely Dispersed*

TABLE 27

THE RELATION BETWEEN (1) THE SCORE MADE BY AN INDIVIDUAL IN CANCELLING WORDS CONTAINING *a* AND *t* AND (2) HIS SCORE IN CANCELLING *A*'S.

Each individual is represented in the table by a pair of values. Thus the first individual scored 10 *in the a-t test and* 36 *in the A test; the second individual scored* 10 *in the a-t test and* 51 *in the A test, etc., etc.*

a-t Words Marked	*A*'s Marked	*a-t* Words Marked	*A*'s Marked	*a-t* Words Marked	*A*'s Marked	*a-t* Words Marked	*A*'s Marked
10	36	17	47	20	58	23	62
10	51	17	49	20	60	23	65
11	43	17	57	20	61	23	70
11	47	18	41	20	62	24	55
11	56	18	43	20	64	24	55
12	45	18	46	20	76	24	59
12	46	18	47	21	45	24	78
13	52	18	47	21	46	25	49
13	55	18	51	21	47	25	54
14	48	18	51	21	48	25	59
14	58	18	53	21	49	25	70
15	37	18	62	21	50	25	78
15	38	18	62	21	54	25	81
15	42	18	63	21	54	26	57
15	43	18	66	21	57	26	60
15	47	19	57	21	59	27	61
15	50	19	60	21	59	27	64
15	52	19	61	21	61	27	65
15	64	19	64	21	63	27	67
15	72	20	38	21	65	27	74
16	43	20	43	22	47	27	78
16	46	20	45	22	48	28	54
16	46	20	46	22	53	28	65
16	55	20	48	22	59	28	65
16	56	20	50	22	62	29	69
16	67	20	51	22	62	30	49
16	70	20	52	22	63	30	59
17	39	20	56	22	77	30	81
17	42	20	56	23	45	34	73
17	44	20	56	23	48		
17	45	20	57	23	58		

Consider Table 27, which shows the relation between (1) the number of words containing both *a* and *t* which an individual marked in a given time and (2) the number of capital A's which the same individual marked in a given time, the same pair of blanks being always used. The table reads:

To mark 10 *a-t* words implied a score of 36 or 51 in the *A* test,
To mark 11 *a-t* words implied a score of 43, 47, or 56 in the *A* test, etc.

The case is identical with Case II., save that the quantities may or may not be on scales with equal units[2] and are not referable to any absolute zero points. Suppose that the units were equal within each of the two scales—that "to mark any one *a-t* word" was equal to "to mark any other," and that "to mark any one *A*" was equal to "to mark any other," but leave the case as it is with regard to the zero points. The relations within the data themselves are then intelligible; we can average the 36 and 51 (getting 43.5), the 43, 47 and 56 (getting 48.7), and so on; we can picture the relation graphically as in Fig. 84. But, the zero points being unknown, we can not refer the relation line to any single pair of axes, or calculate its equation without a w and a z to represent the differences of the arbitrary from the absolute zero points. So the in-

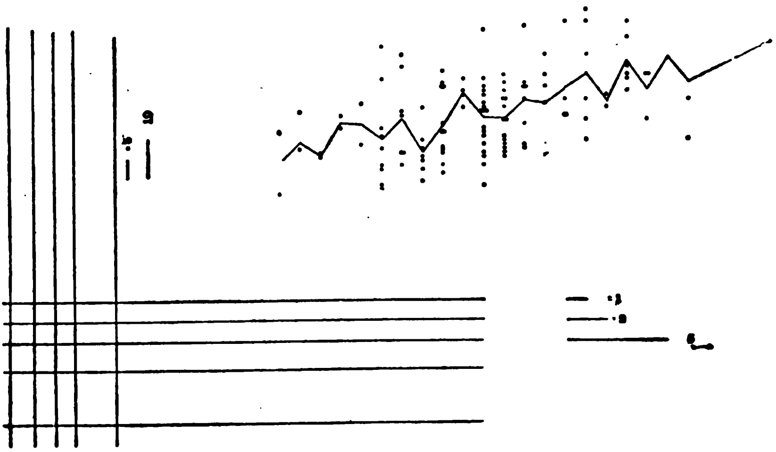

FIG. 84.

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[2] Ten *a-t* words means ten words marked on a certain blank. We can not be sure that the difference between 10 and 12 *a-t* words marked is the same as that between 12 and 14.

definiteness of the vertical and horizontal axes through the absolute zero for both scales have to be represented by leaving such axes out of the graph, or by putting in an indefinite number of them, as is done in Fig. 84.

TABLE 28

a-t Words	*A*'s	*a-t* Words	*A*'s	*a-t* Words	*A*'s	*a-t* Words	*A*'s	*a-t* Words	*A*'s
−10	−19	−3	−16	0	− 7	+1	+10	+ 7	+ 6
−10	− 4	−3	−13	0	− 5	+2	− 8	+ 7	+ 9
− 9	−12	−3	−11	0	− 4	+2	− 7	+ 7	+10
− 9	− 8	−3	−10	0	− 3	+2	− 2	+ 7	+12
− 9	+ 1	−3	− 8	0	+ 1	+2	+ 4	+ 7	+19
− 8	−10	−3	− 6	0	+ 1	+2	+ 7	+ 7	+23
− 8	− 9	−3	+ 2	0	+ 1	+2	+ 7	+ 8	− 1
− 7	− 3	−2	−14	0	+ 2	+2	+ 8	+ 8	+10
− 7	0	−2	−12	0	+ 3	+2	+22	+ 8	+10
− 6	− 7	−2	− 9	0	+ 5	+3	−10	+ 9	+14
− 6	+ 3	−2	− 8	0	+ 6	+3	− 7	+10	− 6
− 5	−18	−2	− 8	0	+ 7	+3	+ 3	+10	+ 4
− 5	−17	−2	− 4	0	+ 9	+3	+ 7	+10	+26
− 5	−13	−2	− 4	0	+21	+3	+10	+14	+18
− 5	−12	−2	− 2	+1	−10	+3	+15		
− 5	− 8	−2	+ 7	+1	− 9	+4	0		
− 5	− 5	−2	+ 7	+1	− 8	+4	0		
− 5	− 3	−2	+ 8	+1	− 7	+4	+ 4		
− 5	+ 9	−2	+11	+1	− 6	+4	+23		
− 5	+17	−1	+ 2	+1	− 5	+5	− 6		
− 4	−12	−1	+ 5	+1	− 1	+5	− 1		
− 4	− 9	−1	+ 6	+1	− 1	+5	+ 4		
− 4	− 9	−1	+ 9	+1	+ 2	+5	+15		
− 4	0	0	−17	+1	+ 4	+5	+23		
− 4	+ 1	0	−12	+1	+ 4	+5	+26		
− 4	+12	0	−10	+1	+ 6	+6	+ 2		
− 4	+13	0	− 9	+1	+ 8	+6	+ 5		

Absolute zero points being unknown, arbitrary ones may be chosen. Thus, each individual may be recorded in the *a-t* test as so much above or below the central tendency of the group (20 *a* − *t* words marked); and in the *A* test, similarly, as + or − from 55 *A*'s marked. The original pairs of values of Table 27 when thus referred to the central tendencies of the two traits as points of reference, become the facts of Table 28 and Fig. 85. When, in place of the varying correspondents of any given value in the *a-t* test,[3] we put their central tendency, we have the facts of Table 29 and Fig. 86. That is, we have evaded the difficulty in respect to zero points by choosing such arbitrarily; and have evaded the difficulty of the dispersion of the values of *B* corresponding to any one value of *A* by taking their central tendency. The problem is reduced to the same problem as in Case I., except for the fact that zero means, in both *A* and *B*, not "*just not any of the thing in question*," but "*the average amount of it.*"

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[3] The varying correspondents in *B* for any given value of *A* are called an *Array*.

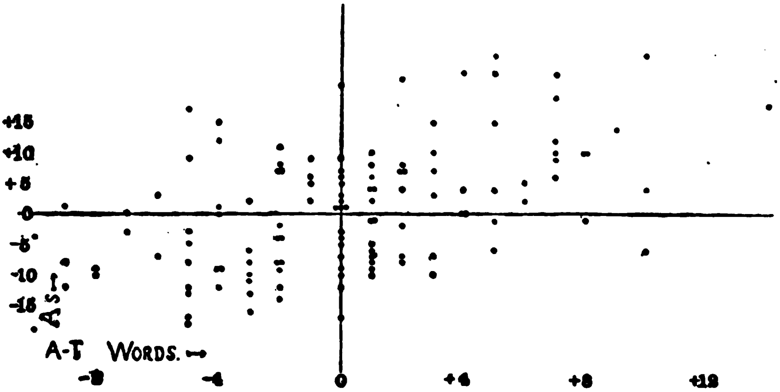

FIG. 85.

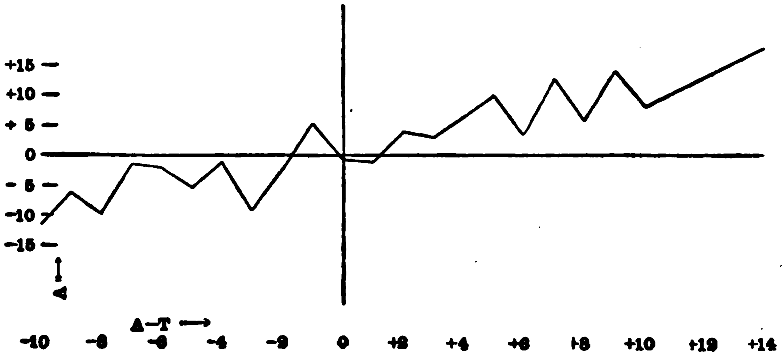

FIG. 86.

TABLE 29

THE RELATION BETWEEN ABILITY IN THE *a-t* TEST AND ABILITY IN THE *A* TEST: THE CENTRAL TENDENCY (AV.) OF THE MEASURES IN THE *A* TEST WHICH ARE RELATED TO EACH VALUE IN THE *a-t* TEST. BOTH SERIES OF VALUES ARE EXPRESSED AS DIVERGENCES FROM THE APPROXIMATE AVERAGE ABILITY OF GIRLS OF GRADE 7B IN A CERTAIN SCHOOL

Ability in the *a-t* Test	Average of the Related Series in the *A* Test
−10	−11.6
− 9	− 6.3
− 8	− 9.5
− 7	− 1.5
− 6	− 2.0
− 5	− 5.55
− 4	− .6
− 3	− 8.8
− 2	− 2.3
− 1	+ 5.5
− 0	− .6
+ 1	− .9
+ 2	+ 3.9
+ 3	+ 3.0
+ 4	+ 6.75
+ 5	+10.2
+ 6	+ 3.5
+ 7	+13.2
+ 8	+ 6.3
+ 9	+14.0
+10	+ 8.0
+14	+18.0

The points of reference could be taken, not as the central tendencies of the two groups of measures whose pairing in a certain way gives the relation in question, but as any two defined points. Thus, in the case of the "*a-t* words-*A*" relation, we can ask what the direction and amount of divergence of an individual from *10 a-t words marked* implies about the direction and amount of his divergence from *36 A's marked.* That is, we can use the lowest record in each case. Or we could take the divergences from 5 and 30, or from 5 and 50, or from any defined points.

§ 33. *The Relation between the Central Tendency of the Values of B Corresponding to Any Given Value of A and that Value of A*

Call the values of *B* which are to be related to any given value of *A*, that value's "Array of *B*'s."

Call a series of values of *A* progressing by equal steps the "*A* Scale."

Call the series of measures which are, in order, the central tendencies of the arrays of *B*'s for successive values of the *A* scale, the "Related Central *B*'s."

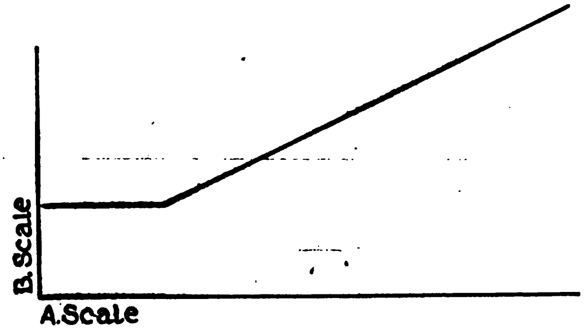

FIG. 87. The Central Relation Line in a relationship where values of *A* below a certain amount make no difference in the central tendency of the related values of *B*.

Call the line which joins the points which represent graphically the Related Central *B*'s, the "Central Relation Line," or simply the "Relation Line."

The Central Relation Line may conceivably take any form. For example, it might be that an increase in *A* up to a certain amount would make no difference in the C.T.'s of the related arrays of *B*, but beyond that amount would imply a steady increase in them. Such a case is shown in Fig. 87. Or it might be that an

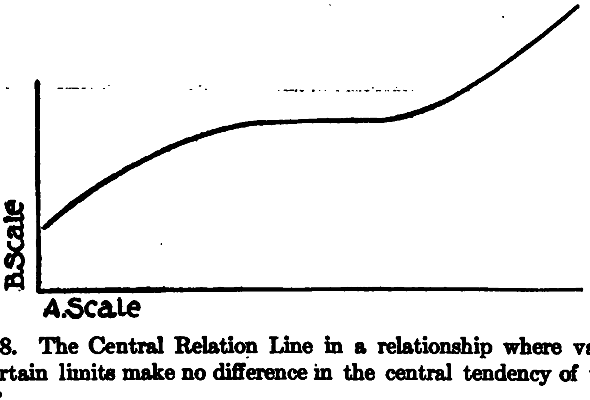

FIG. 88. The Central Relation Line in a relationship where values of *A* between certain limits make no difference in the central tendency of the related values of *B*.

increase in A would, at the low end and high end of the A scale, imply an increase in the C.T.'s of the related arrays of B, but in the

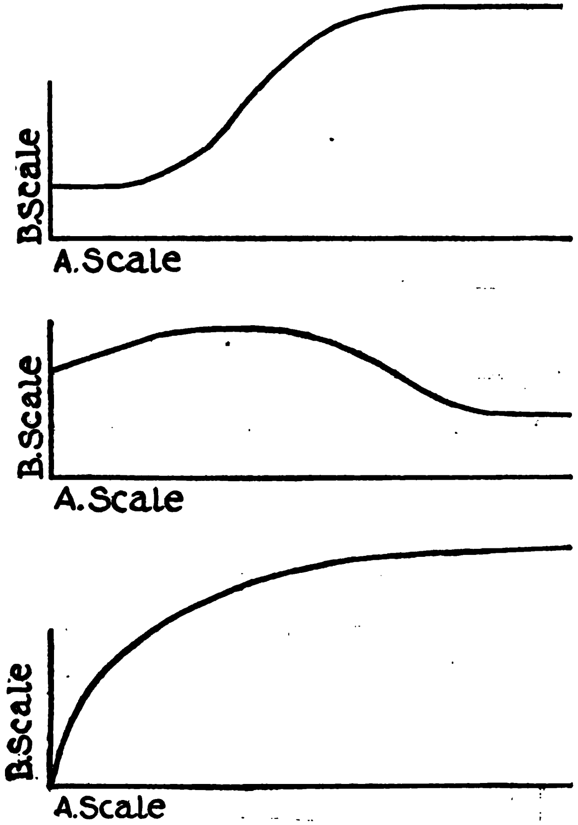

Fig. 89. Samples of possible varieties of Central Relation Lines.

middle range would make no difference. Such a case is shown in Fig. 88. Other possible conditions of the central relation line are shown in Fig. 89.

One of these cases is of special importance, namely the case when the central relation line is straight, its equation being $y - k_1 = C(x - k_2)$, in which C is a measure of the slope of the line, and k_1 and k_2 are constants which are determined by the nature of the zero points for B and A. In such a case of a rectilinear relation, if the central tendencies for A and B are used as the zero points the equation of the line becomes $y = Cx$, and the relation of a related central B to its correspondent in the A scale is uniform throughout. One quantity, denoting the slope of the line, then prophesies what the central related B will be for any given A. For instance the A scale being "20, 21, 22, etc., up to 40," the central related B's being 64, 66, 68, 70, etc., up to 104, the C.T. for A being 30, and the C.T. for B being 84, the relation line is rectilinear, being expressed by the equation $(y - 64) = 2(x - 20)$. When A and B values are

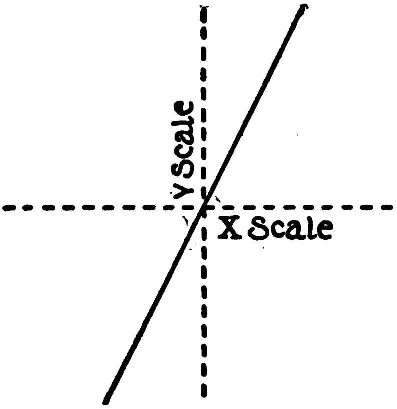

Fig. 90.

referred to their respective C.T.'s, we have as the A scale and its central related B's:

A scale:	−10	− 9	− 8	− 7........+10
Cent. rel. B's:	−20	−18	−16	−14........+20

The equation of the relation line is then $y = 2x$, as shown in Fig. 90.

§ 34. *The Variation in the Values of B Corresponding to Any Given Value of A*

Consider now the separate measures of B in the case where the relation line is rectilinear and where the axes of reference are so chosen that, for the relation line, $y = Cx$. The ratio y/x is then a constant in the case of the central related y's. The ratio of any B value to the A value to which it is related will then be a variable fact, but with C as its central tendency. If all the related pairs are expressed as ratios, the central tendency of these ratios will be the same as the tendency of the central relation line, and will measure the relation of the central tendencies of the arrays. The dispersion of the ratios, each expressing y/x for one pair of values, will measure the variation of the individual relations from C, their central tendency.

If then a relation can be assumed to be rectilinear, its amount—that is, the value of C—can be stated in respect to both the general drift (or the central tendency) of the relation and the amount of departure from that drift or the variability in the relation.

Suppose, for example, that we have, as the related pairs, the facts of Table 30, and have a right to assume that the relation line would, with enough cases, be rectilinear. The twenty B/A ratios, arranged in order of magnitude, are:

−1.00, −1.00, −.33,
1.14, 1.18, 1.80,
2.14, 2.14, 2.15, 2.22, 2.33, 2.67, 2.80,
3.00, 3.00, 3.00, 3.20,
4.00,
5.00, and 5.00.

The median B/A ratio is 2.275; the Q of the ratios is (1.49 + 3.00)/2, or 2.245.

TABLE 30

Value of A	Related Value of B	B/A
−15	−40	2.67
− 9	−20	2.22
− 7	−15	2.14
− 5	−16	3.20
− 5	−14	2.80
− 3	− 9	3.00
− 3	+ 1	−.33
− 1	− 5	5.00
− 1	+ 1	−1.00
− 1	− 5	5.00
+ 1	− 1	−1.00
+ 1	+ 4	4.00
+ 1	+ 3	3.00
+ 3	+ 7	2.33
+ 3	+ 9	3.00
+ 5	+ 9	1.80
+ 7	+ 8	1.14
+ 7	+15	2.14
+11	+13	1.18
+13	+28	2.15

PROBLEMS

36. Name two or three relations that belong under Case II.

37. Name two or three relations that belong under Case III.

38. Name two or three relations that belong under Case IV.

39. The relation to be measured being that between (*A*) the speed at which a person does certain work, say addition, and (*B*) the accuracy with which he works, what would you use as scales for *A* and for *B*, and what would you take in each case as the zero point?

40. Find the median and the *Q* of the relation *A*/*B*, using the facts of Table 30.

CHAPTER XI

CORRELATION

§ 35. *The Problem of Correlation or Mutual Implication*

THE discussion of the preceding chapter was straightforward and in continuity with the procedure in measuring relationships which is familiar to common sense and the sciences in general. For perhaps ninety-nine out of a hundred of the relations which the mental and social sciences need to measure, the simple treatment so far described suffices, the precautions necessary being to face frankly the great variability of the relations (that is, the great dispersion of the measures of B related to any given value of A), and to keep in mind the meaning of the arbitrary zero points chosen in all statements of the relation and in all inferences from it. But under certain circumstances radically different methods of measuring a relation need to be employed, and these methods, though of essentially minor importance in the mental and social sciences generally, require rather elaborate explanation.

The chief circumstances which make their use desirable are: first, the need of exact measurement of the peculiar relation of *likeness, resemblance, correspondence in magnitude;* and second, the need of *comparing quantitatively two or more relations* of this peculiar sort. For example, in studies of heredity, one needs to measure the resemblance between sons and fathers, between sons and grandfathers, and between sons and any other males taken at random from the same race; and to compare these three resemblances quantitatively. So, also, in studies of educational or industrial diagnosis, one needs to measure the resemblance between boys' total intelligence and capability and their achievement in school, between the former and their achievement in a certain set of mental tests, and between one and another of various further facts about them. Here also one needs to compare, say, the amount of resemblance between "total intellect" and "school record" with the amount of resemblance between "total intellect" and "record in the mental tests."

These two needs have been met by various methods of measuring the mutual implication, or *correlation*, of paired values of A and B. The correlation between A and B is neither the relation of B to A nor that of B to A, but a peculiar composite of certain elements of both relations taken together. Just what a correlation is can be seen best by observing just what the different measures of correlation do measure.

§ 36. *The Data Available for Estimating Correlation*

Similarity in Relative Position.—Suppose that we have for ten boys the following measures:

	Traits			
Boy	*A* Total Intellect	*B* School Achievement	*C* Score in Mental Tests	*D* Score in Drawing
a	30	46	15	78
b	32	40	16	83
c	36	58	18	79
d	38	61	18	84
e	39	66	19	75
f	42	57	21	78
g	45	67	22	81
h	47	55	24	86
i	47	85	23	76
j	52	70	26	82

Consider the three correlations—B with A, C with A and D with A—in respect to the question, "How far do the two series of pairs to be related correspond, in respect to order?"

The orders in the four cases are:

Boy	*A*	*B*	*C*	*D*
a	1	2	1	3.5
b	2	1	2	8
c	3	5	3.5	5
d	4	6	3.5	9
e	5	7	5	1
f	6	4	6	3.5
g	7	8	7	6
h	8	3	8	10
i	9	10	9	2
j	10	9	10	7

The differences between the orders are:

For B and A the sum of the differences[1] $(\Sigma D_{BA}) = 18$,
For C and A the sum of the differences $(\Sigma D_{CA}) = 1$,
For D and A the sum of the differences $(\Sigma D_{DA}) = 35$.

The sum of the differences in a series of related pairs evidently tells something about the mutual implication or correlation and gives some aid in comparing one correlation with another quantitatively.

Similarity in Direction (+ or −) from the Points of Reference.—Suppose that the arbitrary zero points, to which the facts to be related are to be referred, are 40, 60, 20 and 80 for *A*, *B*, *C* and *D*, respectively. Then the measures, so referred, become those of Table 31.

TABLE 31

	Traits			
Boy	*A*	*B*	*C*	*D*
a	−10	−14	−5	−2
b	− 8	−20	−4	+3
c	− 4	− 2	−2	−1
d	− 2	+ 1	−2	+4
e	− 1	+ 6	−1	−5
f	+ 2	− 3	+1	−2
g	+ 5	+ 7	+2	+1
h	+ 7	− 5	+4	+6
i	+ 7	+25	+3	−4
j	+12	+10	+6	+2

Consider the same three correlations—*B* with *A*, *C* with *A* and *D* with *A*—in respect to the question, "How often does + go with + and − with −?"

The percentages of like-signed pairs are 60, 100 and 60. The percentage of like-signed pairs evidently tells something about a relation and gives some aid in comparing one relation with another quantitatively—*C* being, in so far forth, shown to be more closely related to *A* than *B* or *D* is.

The Measures Expressed as Multiples of the Variability of the Trait in Question in the Group in Question.—Consider the same three relations in respect to the question, "What are the measures themselves and the ratios, B/A and A/B, C/A and A/C, and D/A and A/D, when each measure in Table 31 is expressed as a multiple of the variability of the group in question in the trait in question?"

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[1] Regardless of signs.

Suppose the variability of the group in question to be, in *A*, *B*, *C* and *D*, respectively, 6, 9, 3 and 3 (using the A.D. as the measure of variability in each case).[2] Then the measures of Table 31 become those of Table 32.

TABLE 32

The Facts of Table 31, Each Expressed as a Multiple of the Variability of the Trait in Question in the Group in Question

Boy	*A*	*B*	*C*	*D*
a	−1.67 Var. *A*	−1.56 Var. *B*	−1.67 Var. *C*	− .67 Var. *D*
b	−1.33 Var. *A*	−2.22 Var. *B*	−1.33 Var. *C*	+1.00 Var. *D*
c	− .67 Var. *A*	− .22 Var. *B*	− .67 Var. *C*	− .33 Var. *D*
d	− .33 Var. *A*	+ .11 Var. *B*	− .67 Var. *C*	+1.33 Var. *D*
e	− .17 Var. *A*	+ .67 Var. *B*	− .33 Var. *C*	−1.67 Var. *D*
f	+ .33 Var. *A*	− .33 Var. *B*	+ .33 Var. *C*	− .67 Var. *D*
g	+ .83 Var. *A*	+ .78 Var. *B*	+ .67 Var. *C*	+ .33 Var. *D*
h	+1.17 Var. *A*	− .56 Var. *B*	+1.33 Var. *C*	+2.00 Var. *D*
i	+1.17 Var. *A*	+2.78 Var. *B*	+1.00 Var. *C*	−1.33 Var. *D*
j	+2.00 Var. *A*	+1.11 Var. *B*	+2.00 Var. *C*	+ .67 Var. *D*

Mere inspection of the measures, each expressed as a multiple of the variability of the group in question in the trait in question, shows that the individuals now almost duplicate in *C* their measures in *A*; that they show little resemblance, each in *D* with himself in *A*; and that the correlation of *B* with *A* is intermediate. The calculation of the ratios *B/A*, *A/B*, *C/A*, *A/C*, *D/A* and *A/D* gives the results shown in Table 33.

TABLE 33

Ratios Computed from the Facts of Table 32

Boy	*B/A*	*A/B*	*C/A*	*A/C*	*D/A*	*A/D*
a	.93	1.07	1.00	1.00	.40	2.50
b	1.67	.60	1.00	1.00	neg.	neg.
c	.33	3.00	1.00	1.00	.50	2.00
d	neg.	neg.	2.00	.50	neg.	neg.
e	neg.	neg.	2.00	.50	10 00	.10
f	neg.	neg.	1 00	1.00	neg.	neg.
g	.93	1.07	.80	1.25	.40	2.50
h	neg.	neg.	1.14	.88	1.71	.58
i	2.38	.42	.86	1.17	neg.	neg.
j	.56	1.80	1.00	1.00	.33	3.00

[2] The A.D.'s of the ten boys here listed vary slightly from these figures.

The Use of Ratios.—The median of the B/A ratios is + .45; that of the A/B ratios is +.51; that of the C/A ratios is + 1.00; that of the A/C ratios is + 1.00; that of the D/A ratios is + .37; that of the A/D ratios is + .34.

The median of the B/A and A/B ratios together is .49; that of the C/A and A/C ratios together is 1.00; that of the D/A and A/D ratios together is .37.

The use of the B/A and A/B ratios to estimate correlation enables us to define correlation in terms of the more straightforward relations described in the previous chapter. Assume a relation to be rectilinear; express the A's and B's as such divergences that the equation of the relation line is $y = Cx$; express each A as so many times the A.D. (or σ or Q) of the A's and each B as so many times the A.D (or σ or Q) of the B's. Then the central tendency of the relation of B to A—the slope of the relation line—will be, by the method of the previous chapter, .45. In the same way the central tendency of the relation of A to B will be .51. In the same way the mutual relation—of B to A and A to B, taken together—will be .49. This mutual relation *is* the correlation.

A correlation is a *mutual*, not a one-direction, relation; is not the relation of absolute amounts of divergence, but is the relation of such amounts *divided by the variability of the trait in question;* and assumes, in so far as a single coefficient is to be its adequate measure, that the relation lines for A to B and B to A are rectilinear.

TABLE 34

DIFFERENCES COMPUTED FROM THE FACTS OF TABLE 32

Boy	$B-A$	$C-A$	$D-A$
a	.11	0	1.00
b	.89	0	2.33
c	.45	0	.33
d	.44	.33	1.67
e	.84	.17	1.50
f	.66	0	1.00
g	.05	.17	.50
h	1.73	.17	.83
i	1.61	.17	2.50
j	.89	0	1.33
ΣD	7.67	1.01	12.99

The Use of Differences.—If, instead of the ratios, B/A, C/A, D/A, we use the differences, $B - A$, $C - A$, and $D - A$, we have (A, B, C and D being expressed as deviations from 40, 60, 20 and 80 respectively and as multiples of 6, 9, 3 and 3 respectively) the facts of Table 34.

The sums of these differences (their signs being disregarded) are: for $B - A$, 7.67; for $C - A$, 1.01; and for $D - A$, 12.99.

The Use of Products.—If, instead of the ratios, B/A, C/A and D/A, we take the products—$B \times A$, $C \times A$, and $D \times A$—using the facts of Table 32 as before, we have the facts of Table 35.

TABLE 35

PRODUCTS COMPUTED FROM THE FACTS OF TABLE 32

Boy	$B \times A$		$C \times A$	$D \times A$	
a	2.61		2.79	1.12	
b	2.95		1.77		−1.33
c	.15		.45	.22	
d		−.04	.22		− .44
e		−.11	.06	.28	
f		−.10	.10		− .22
g	.65		.56	.27	
h		−.66	1.56	2.34	
i	3.25		1.17		−1.56
j	2.22		4.00	1.33	
	$\Sigma B \cdot A = 10.92$		$\Sigma C \cdot A = 12.68$	$\Sigma D \cdot A = 2.01$	

The sum of the $C \times A$ products is 12.68, while the sum of the $D \times A$ products is only 2.01.

In the ratios B/A, A/B, C/A, A/C, D/A and A/D or in the differences $B - A$, $C - A$ and $D - A$, or in the products $B \times A$, $C \times A$, and $D \times A$, the original A's, B's, C's, and D's having been in each case expressed as deviation-measures and divided by the variability, there are means of comparing one relation with another quantitatively.

The agreement of the two values of any related pair, when each is replaced by a number denoting its relative position in an order of magnitude; the percentage which the $++$ and $--$ pairs are of the total number of pairs, when each of the two values of a related pair is expressed as a difference $+$ or $-$ from a defined point of reference; and, when each is so expressed and also turned into a multiple of the variability of the group in question in the trait in question, the ratios B/A and A/B, the differences $B - A$ or $A - B$ (regardless of signs), and the AB products: these are some of the facts about a series of related pairs which are used to obtain a quantitative account of the correlation.

§ 37. *Coefficients of Correlation*

Some of the formulæ by which they are so used are the following, r being always the measure of the correlation:

I.

$$\rho = 1 - \frac{6\Sigma D^2}{n(n^2 - 1)}$$

in which ΣD^2 = the sum of the squares of the differences between the two numbers denoting the relative positions of the two related measures in their respective series; and n = the number of pairs of related measures.

$$r_{\text{I}} = 2 \sin \left(\frac{\pi}{6}\rho\right)$$

II. The measures to be related being expressed as divergences from defined points of reference—the A's from the central tendency of the A's and the B's from the central tendency of the B's.

$$r_{\text{II}} = \text{cosine}\,\pi U$$

in which U = the proportion which the number of unlike signed pairs is of the total number of pairs, when every measure is given its + or − sign of divergence.

III*a*. r = the median of the A/B and B/A ratios, the A's and B's being expressed as divergences from defined points of reference, in multiples of the variability of the A's and of the variability of the B's respectively. That is, $r_{\text{III}a}$ = the median of the ratios

$$\frac{\frac{A_1}{\sigma_A}}{\frac{B_1}{\sigma_B}},\ \frac{\frac{B_1}{\sigma_B}}{\frac{A_1}{\sigma_A}},\ \frac{\frac{A_2}{\sigma_A}}{\frac{B_2}{\sigma_B}},\ \frac{\frac{B_2}{\sigma_B}}{\frac{A_2}{\sigma_A}},\ \cdots\ \frac{\frac{A_n}{\sigma_A}}{\frac{B_n}{\sigma_B}},\ \frac{\frac{B_n}{\sigma_B}}{\frac{A_n}{\sigma_A}}.$$

III*b*. The A's and B's being expressed as in III*a*

$$r_{\text{III}b} = \frac{\Sigma\left(\frac{A}{\sigma_A}\right)\left(\frac{B}{\sigma_B}\right)}{\sqrt{\Sigma\left(\frac{A}{\sigma_A}\right)^2}\sqrt{\Sigma\left(\frac{B}{\sigma_B}\right)^2}},$$

where $\Sigma(A/\sigma_A)(B/\sigma_B)$ = the sum of the products of each related pair of values, the A's and B's being expressed as multiples of the variability of the A's and of the variability of the B's, respectively; $\Sigma(A/\sigma_A)^2$ = the sum of the squares of the A's similarly expressed; and $\Sigma(B/\sigma_B)^2$ = the sum of the squares of the B s similarly expressed.

In III*a* any other measure of variability may replace σ provided the same substitution occurs throughout. So also in III*b*.

The reader should note that in all these formulæ the gross amounts of the values to be related no longer figure. In I., amounts are replaced by relative positions. In II., each amount is replaced by the mere direction of the divergence plus or minus. In III., the amounts are divided through by the variability of the trait in the group, so that a pair, such as "$A = 1{,}004$ *related to* $B = 28$," means "$+\ 2$ *related to* $+\ 2$" if the points of reference are 1,000 and 20, and the variabilities of the A's and B's are 2 and 4, respectively.

The reader should note further that the maximum for r by any of the three sets of formulæ is 1, and that its minimum is $-\ 1$. Thus, in I., if the two series of relative positions agree perfectly as paired in the relation in question, $\Sigma D^2 = 0$. If the ΣD^2 is the greatest possible, it is twice $n(n^2 - 1)/6$ (*e. g.*, for a series of 10 pairs we have $81 + 49 + 25 + 9 + 1 + 1 + 9 + 25 + 49 + 81 = 330$ and $10.99/6 = 165$). In II., if $U = 0$ we have $r = \cos 0$ which $= 1$. To get negative values of r—values corresponding to values of U from .50 to 1.00—the sense of the formula and the signs of r are reversed together. This the reader may accept for the present without justification. In III*a*, a series of pairs showing the most perfect correlation—such as $-\ 8 -\ 4$, $-\ 6 -\ 3$, $-\ 4 -\ 2$, $-\ 2 -\ 1$, $+\ 2 +\ 1$, $+\ 4 +\ 2$, $+\ 6 +\ 3$ and $+\ 8 +\ 4$—will give a median mutual ratio of 1 when each value is expressed as a multiple of the variability of its series; and the most antagonistic relation will give such a ratio of $-\ 1$. The r's got by the formula (III*b*) using the products will be found similarly to be 1.00 and $-\ 1.00$ for the greatest and least correlation.

Note, in the third place, that when the correlation is such as would come if the values to be related were paired at random, $r = 0$. Thus, in I., the sum of the squares of the differences from pairing at random a series of numbers 1, 2, 3 . . . n with another identical series will be $n(n^2 - 1)/6$, and r will equal $1 - 1$. In II., random pairing will give, obviously, one half of the pairs as unlike-signed and $r = \cos \pi \cdot \frac{1}{2}$, or $r = 0$. In III*a*, half of the ratios will be negative in random pairing. In III*b*, the sum of the negative products will equal the sum of the positive products.

A correlation, as measured by one of these formulæ, then, utilizes, in the case of any measure of a pair, its position relative to others in the series, or the direction only of its divergence from the point of reference, or the amount of its divergence from that point divided by the variability of the series—that is, the general tendency of the series to diverge from that point. It results in a measure varying from -1.00 through 0—or what the fact would be by random pairing—to $+1.00$. It measures the slope of a relation line which is a compromise of the slopes

$$\frac{\frac{y}{\sigma_y}}{\frac{x}{\sigma_x}} \text{ and } \frac{\frac{x}{\sigma_x}}{\frac{y}{\sigma_y}};$$

or, using different words for the same fact, it measures a ratio which is a compromise between the central tendency of the

$$\frac{\frac{y}{\sigma_y}}{\frac{x}{\sigma_x}} \text{ ratios and the central tendency of the } \frac{\frac{x}{\sigma_x}}{\frac{y}{\sigma_y}} \text{ ratios.}$$

§ 38. *The Comparability of Coefficients of Correlation*

Comparability of r's Got from the Same Set of Pairs by the Different Methods.—Provided (1) that the point of reference taken for the *A*'s is, by each method, taken as the central tendency of the same group, *N*, and (2) that the point of reference taken for the *B*'s is also taken, by each method, as the central tendency of the same group, *N*, methods II., III*a*, and III*b* give comparable results, the r in each case measuring approximately the same fact. The r

by method I. also, if the values of A and the values of B are in each case distributed in a surface of frequency of approximately Form A without large or frequent gaps, is roughly equal to the corresponding r's by II. and III.

In each case the r represents an inference about the probable general drift of the relation, supposing it to be rectilinear. Hence all are comparable, if the above conditions are fulfilled. The data used in making the inference are, however, different according to the method, and the three r's got for the same series of pairs from methods II., III*a* and III*b* are, in strictness, no more absolutely interchangeable than an average, a median and a mode got from the same series of single values would be. An r got by method I. is still less absolutely comparable with any one of the other three r's.

Comparability of r's Got from two Different Sets of Pairs by the Same Method.—Suppose the correlation of A with B and the correlation of C with D to have been measured, in each case by the same method, r being found to be, say .4 in each case. The two correlations may be said to be equal in the following sense: (1) correlation to mean the general drift of a relation-line; (2) distance of an A from the point of reference taken for the A's divided by the σ[1] of the A's for group N, to be assumed as equal to the numerically equal value got by dividing the distance of a B from the point of reference taken for the B's by the σ[1] of the B's for the group.

Suppose that, the same method I., II., III*a* or III*b* being used, the r_{AB} for A and B and the r_{CD} for C and D come out, each as the same numerical value, say .4 or .63 or $-$ .272. There is then a tendency for the student to assume that the relation of A to B is identical with that of C to D. But the numerical identity of the *correlations* alone proves nothing about the real identity of the *relations*. The r must in each case be interpreted in the light of the measures from which it is derived. Consider, for example, two such r's got by method III*a*. The numbers used to get the r's were (call them the x's) the result of dividing the A deviation measures by some number (call it *var. A in N*) expressing the variability of trait A in some group, the B deviation measures (call them the y's) by some number (call it *var. B in N*) expressing the variability of the trait B in some group, etc. Now equality of r_{AB} and r_{CD}, both being .4, depends on the assumption that any $\frac{x}{\text{var. } A \text{ in } N'}$, any $\frac{y}{\text{var. } B \text{ in } N_2}$, any $\frac{z}{\text{var. } C \text{ in } N_3}$ and any $\frac{w}{\text{var. } D \text{ in } N_4}$ are equal in fact if they are numerically equal. This assumption must be kept in mind.

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[1] Or any other measure of variability consistently used.

Further, the x's, y's, z's, and w's were the results of calculating divergences from four points of reference, A_0, B_0, C_0, and D_0. Any inference of identity later in the process depends upon the assumption that A_0, B_0, C_0, and D_0 have identical values as points of reference. Finally we have the fundamental limitation that r measures only the general drift of the relation.

Similar need for interpretation will be seen to hold good for any of the other methods. The student should therefore always make comparison of any two relations only after thinking of them in terms of the actual facts which they report. It is desirable for the beginner to use some such systematic form as:

For Method I.:

The correlation AB measures the general closeness of correspondence between (1) the relative positions of................
in trait ....
and (2) the relative positions of............................
in trait ....
The relative positions being paired after the fashion..........
....

For Methods II. and III.:

The correlation AB measures the general drift of the relation between (1) the divergences of...........................
in the trait...
from.......................................as a point of
reference, each divergence being expressed as a multiple of......
....
and (2) divergences of...................................
in the trait...
from.......................................as a point of
reference, each divergence being expressed as a multiple of......
....

The divergences being paired after the fashion..............

....

The method of estimating the relation being by..................
(State whether it is by the percentage of unlike signed pairs, or by the median ratio, or by the percentage which the sum of the pair-products is of the sum of the maximal pair-products.)

§ 39. *The Technique of Measuring Correlations*

Throughout this section, the following symbols will be used:

Call the original paired values to be correlated A_1 and B_1, A_2 and B_2 . . . A_n and B_n.

Call the divergences of A_1, A_2 . . . A_n from the C.T. of the A's x_1, x_2 . . . x_n.

Call the divergences of B_1, B_2 . . . B_n from the C.T. of the B's y_1, y_2 . . . y_n.

Let x = any one of the series x_1, x_2 . . . x_n.

Let y = any one of the series y_1, y_2 . . . y_n.

Let x/y = any one of the series x_1/y_1, x_2/y_2 . . . x_n/y_n.

Let y/x = any one of the series y_1/x_1, y_2/x_2 . . . y_n/x_n.

Let σ_A, A.D.$_A$ and Q_A be the σ, A.D. and Q of the A's.

Let σ_B, A.D.$_B$ and Q_B be the σ, A.D. and Q of the B's.

By Differences in Relative Positions or Ranks.—The formulæ are

$$r = 2 \sin\left(\frac{\pi}{6}\rho\right) \quad \text{and } \rho = 1 - \frac{\Sigma D^2}{\frac{n(n^2-1)}{6}}.$$

The formula

$$\rho = 1 - \frac{\Sigma D^2}{\frac{n(n^2-1)}{6}}$$

needs no comment save with respect to the positions to be assigned to two or more identical amounts of A (or of B). The rule is to keep the largest position-number in the case of both A and B equal to the number of pairs. So, identical amounts are each given, as a position-number, the average of the positions which they would take were they slightly different. Thus, suppose the series of amounts of A to be 20, 21, 22, 22, 23, 23, 23, 24, 24, 24, 24. The corresponding position-numbers given would be 1, 2, 3.5, 3.5, 5, 5, 5, 9.5, 9.5, 9.5, 9.5. The value of r for any given value of ρ may be obtained from Table 36, if the form of distribution is approximately Form *A* for each of the two traits. If the form of distribution is in each case approximately a rectangle r may be taken as equal to ρ.

TABLE 36

A TABLE TO INFER THE VALUE OF r FROM ANY GIVEN VALUE OF ρ.

$$\rho = 1 - \frac{6\Sigma D^2}{n(n^2 - 1)}$$

ρ	r	ρ	r	ρ	r	ρ	r
.01	.0105	.26	.2714	.51	.5277	.76	.7750
.02	.0209	.27	.2818	.52	.5378	.77	.7847
.03	.0314	.28	.2922	.53	.5479	.78	.7943
.04	.0419	.29	.3025	.54	.5580	.79	.8039
.05	.0524	.30	.3129	.55	.5680	.80	.8135
.06	.0628	.31	.3232	.56	.5781	.81	.8230
.07	.0733	.32	.3335	.57	.5881	.82	.8325
.08	.0838	.33	.3439	.58	.5981	.83	.8421
09	.0942	.34	.3542	.59	.6081	.84	.8516
.10	.1047	.35	.3645	.60	.6180	.85	.8610
.11	.1151	.36	.3748	.61	.6280	.86	.8705
.12	.1256	.37	.3850	.62	.6379	.87	.8799
.13	.1360	.38	.3935	.63	.6478	.88	.8893
.14	.1465	.39	.4056	.64	.6577	.89	.8986
.15	.1569	.40	.4158	.65	.6676	.90	.9080
.16	.1674	.41	.4261	.66	.6775	.91	.9173
.17	.1778	.42	.4363	.67	.6873	.92	.9269
.18	.1882	.43	.4465	.68	.6971	.93	.9359
.19	.1986	.44	.4567	.69	.7069	.94	.9451
.20	.2091	.45	.4669	.70	.7167	.95	.9543
.21	.2195	.46	.4771	.71	.7265	.96	.9635
.22	.2299	.47	.4872	.72	.7363	.97	.9727
.23	.2403	.48	.4973	.73	.7460	.98	.9818
.24	.2507	.49	.5075	.74	.7557	.99	.9909
.25	.2611	.50	.5176	.75	.7654	1.00	1.0000

As a still more convenient measure, we may use

$$r = \text{sine}\,\frac{\pi}{2}R$$

or

$$r = 2\,\text{cosine}\,\frac{\pi}{3}(1 - R) - 1,$$

where

$$R = 1 - \frac{6\Sigma g}{n^2 - 1}$$

Σg being the sum of the *plus* differences in ranks.

The formula, $r = 2 \text{ cosine } (\pi/3)(1 - R) - 1$, is correct if the two traits are distributed as in Form *A*. The formula, $r = \text{sine } (\pi/2)R$, was determined empirically as a fair account of the relation between r and R in certain concrete cases, by Spearman, who devised the formulæ for using relative positions in correlation. Table 37 gives the value of r for any given value of R, according to the equation $r = \text{sine } (\pi/2)R$. Table 38 gives the value of r for any given value of R, according to $r = 2 \cos (\pi/3)(1 - R) - 1$.

TABLE 37

A Table to Infer the Value of r from Any Given Value of R, According to $r = \sin (\pi/2)R$. $R = 1 - (6\Sigma G)/(n^2 - 1)$

R	r	R	r	R	r	R	r
.00	.000						
.01	.016	.26	.397	.51	.718	.76	.930
.02	.031	.27	.412	.52	.729	.77	.935
.03	.047	.28	.426	.53	.740	.78	.941
.04	.063	.29	.440	.54	.750	.79	.946
.05	.078	.30	.454	.55	.760	.80	.951
.06	.094	.31	.468	.56	.771	.81	.956
.07	.110	.32	.482	.57	.780	.82	.960
.08	.125	.33	.496	.58	.790	.83	.965
.09	.141	.34	.509	.59	.800	.84	.969
.10	.156	.35	.522	.60	.809	.85	.972
.11	.172	.36	.536	.61	.818	.86	.976
.12	.187	.37	.549	.62	.827	.87	.979
.13	.203	.38	.562	.63	.836	.88	.982
.14	.218	.39	.575	.64	.844	.89	.985
.15	.233	.40	.588	.65	.853	.90	.988
.16	.249	.41	.600	.66	.861	.91	.990
.17	.264	.42	.613	.67	.869	.92	.992
.18	.279	.43	.625	.68	.876	.93	.994
.19	.294	.44	.637	.69	.884	.94	.996
.20	.309	.45	.649	.70	.891	.95	.997
.21	.324	.46	.661	.71	.898	.96	.998
.22	.339	.47	.673	.72	.905	.97	.999
.23	.353	.48	.685	.73	.911	.98	.9995
.24	.368	.49	.696	.74	.918	.99	.99988
.25	.383	.50	.707	.75	.924	1.00	1.000

TABLE 38

A TABLE TO INFER THE VALUE OF r FROM ANY GIVEN VALUE OF R, ACCORDING TO $r = 2\cos\frac{\pi}{3}(1 - R) - 1$. $R = 1 - \frac{6\Sigma G}{n^2 - 1}$

R	r	R	r	R	r	R	r
.00	.000						
.01	.018	.26	.429	.51	.742	.76	.937
.02	.036	.27	.444	.52	.753	.77	.942
.03	.054	.28	.458	.53	.763	.78	.947
.04	.071	.29	.472	.54	.772	.79	.952
.05	.089	.30	.486	.55	.782	.80	.956
.06	.107	.31	.500	.56	.791	.81	.961
.07	.124	.32	.514	.57	.801	.82	.965
.08	.141	.33	.528	.58	.810	.83	.968
.09	.158	.34	.541	.59	.818	.84	.972
.10	.176	.35	.554	.60	.827	.85	.975
.11	.192	.36	.567	.61	.836	.86	.979
.12	.209	.37	.580	.62	.844	.87	.981
.13	.226	.38	.593	.63	.852	.88	.984
.14	.242	.39	.606	.64	.860	.89	.987
.15	.259	.40	.618	.65	.867	.90	.989
.16	.275	.41	.630	.66	.875	.91	.991
.17	.291	.42	.642	.67	.882	.92	.993
.18	.307	.43	.654	.68	.889	.93	.995
.19	.323	.44	.666	.69	.896	.94	.996
.20	.338	.45	.677	.70	.902	.95	.997
.21	.354	.46	.689	.71	.908	.96	.998
.22	.369	.47	.700	.72	.915	.97	.999
.23	.384	.48	.711	.73	.921	.98	.9996
.24	.399	.49	.721	.74	.926	.99	.9999
.25	.414	.50	.732	.75	.932	1.00	1.0000

By Percentage of Unlike Signed Pairs.—The value of r for any given proportion of unlike-signed pairs is conveniently obtained from Table 39.

In using Table 39, that is, in using the formula $r = \cos \pi U$ there should theoretically be no zero values of either x or y. When such values are unavoidable, they may be treated as follows:

Call the total number of pairs, n,
Call the number of + + and − − pairs, l,
Call the number of + − pairs, u,
Call the number of '00,' '+ 0,' '0 +,' '− 0' and '0 −' pairs, d.
Divide the d's between the l's and the u's in a proportion half way between (1) half and half and (2) the proportion in which the l and u pairs stand. That is let

$$U = \frac{u + \left(\dfrac{\dfrac{u}{u+l} + \dfrac{1}{2}}{2}\right)d}{n}$$

This is an arbitrary compromise. Two defensible suppositions can be made. First, at their face value, all pairs containing a zero are as likely to go l as u. Second, with a fine grouping, the zero cases would be more likely to divide up in the same proportion of l and u as characterizes the rest of the pairs than in an equally reverse proportion. Thus if $l = 90$, $u = 10$ and $d = 10$, it seems unlikely that with fine grouping the 10 zero pairs would have as few l's as u's, and very much more likely that they should be 9 l's and 1 u than that they should be 1 l and 9 u's.

TABLE 39

VALUES OF r CORRESPONDING TO EACH PERCENTAGE OF UNLIKE-SIGNED PAIRS. IF THE PERCENTAGES ARE TAKEN AS THOSE OF THE LIKE-SIGNED PAIRS, THE r'S ARE NEGATIVE. r = THE COEFFICIENT OF CORRELATION, U = THE NUMBER OF UNLIKE-SIGNED PAIRS DIVIDED BY THE NUMBER OF LIKE-SIGNED AND UNLIKE-SIGNED PAIRS.

U	r	U	r
.00	1.0000	.26	.6848
.01	.9996	.27	.6615
.02	.9982	.28	.6375
.03	.9958	.29	.6129
.04	.9924	.30	.5877
.05	.9880	.31	.5620
.06	.9826	.32	.5358
.07	.9762	.33	.5091
.08	.9688	.34	.4819
.09	.9604	.35	.4542
.10	.9510	.36	.4260
.11	.9407	.37	.3973
.12	.9295	.38	.3682
.13	.9174	.39	.3387
.14	.9044	.40	.3089
.15	.8905	.41	.2788
.16	.8757	.42	.2485
.17	.8602	.43	2180
.18	.8439	.44	.1873
.19	.8268	.45	.1564
.20	.8089	.46	.1253
.21	.7902	.47	.0941
.22	.7707	.48	.0628
.23	.7504	.49	.0314
.24	.7293	.50	.0000
.25	.7074		

By the Central Tendency of the Ratios.—In calculating the median ratio it is not necessary to divide every value of x by the variability of A and every value of y by the variability of B. Only such calculations need be made as suffice to get the median of the

$$\frac{\frac{x}{\sigma_A}}{\frac{y}{\sigma_B}} \text{ and } \frac{\frac{y}{\sigma_B}}{\frac{x}{\sigma_A}}$$

ratios.[4] Only a few ratios near the median of the gross x/y series and near the median of the gross y/x series need to be divided through.

If the number of pairs is 20 or more the medians of the gross x/y series and of the y/x series will give a sufficiently close approximation for r by the use of the formula:[5]

$$r = \sqrt{[(\text{mid } x/y)v_1][(\text{mid } y/x)v_2]}$$

or, approximately,

$$r = \frac{(\text{mid } x/y)v_1 + (\text{mid } y/x)v_2}{2},$$

in which

$\text{mid } x/y =$ the median of the gross x/y ratios,
$\text{mid } y/x =$ the median of the gross y/x ratios,

$$v_1 = \frac{\sigma_B}{\sigma_A} \text{ or } \frac{\text{A.D.}_B}{\text{A.D.}_A} \text{ or } \frac{Q_B}{Q_A},$$

$$v_2 = \frac{1}{v_1}.$$

By the Sum of the Pair-Products.—In calculating the percentage which the sum of the pair-products is of their maximal sum, we use, in place of the formula

$$r = \frac{\Sigma\left(\frac{x}{\sigma_A}\right)\left(\frac{y}{\sigma_B}\right)}{\sqrt{\Sigma\left(\frac{x}{\sigma_A}\right)^2}\sqrt{\Sigma\left(\frac{y}{\sigma_B}\right)^2}},$$

---

[4] A.D.$_A$ and A.D.$_B$, or Q_A and Q_B, may replace σ_A and σ_B, if consistency in the measure of variability is maintained.

[5] Indeed, the use of this formula is in general preferable to getting the one median for all the $\frac{\frac{x}{\sigma_x}}{\frac{y}{\sigma_y}}$ and $\frac{\frac{y}{\sigma_y}}{\frac{x}{\sigma_x}}$ ratios taken together.

the simpler one

$$r = \frac{\Sigma(x \cdot y)}{\sqrt{\Sigma x^2}\,\sqrt{\Sigma y^2}},$$

since the effect of dividing through by the variability is the same in both numerator and denominator.

This is the Pearson coefficient of correlation, usually stated in the less convenient form,

$$r = \frac{\Sigma(x \cdot y)}{n\sigma_A\sigma_B}.$$

In order to economize time, it is desirable to calculate the x's as divergences from a point at the middle of a step, or just between two steps, of the A scale; and similarly for the y's. From the correlation so calculated, the correlation that would be found if some other points of reference were used in getting the x's and the y's, can be estimated by the following correction:

Let A_a be the point actually used in calculating the x's,
Let B_a be the point actually used in calculating the y's,
Let A_0 and B_0 be the points from which the x's and y's should have been calculated.

Then the correction is to subtract $n(A_0 - A_a)(B_0 - B_a)$ from the numerator in the formula,

$$r = \frac{\Sigma x \cdot y}{\sqrt{\Sigma x^2}\,\sqrt{\Sigma y^2}}.$$

The calculations necessary to obtain r by each of these methods are shown in Tables 40 and 41, which also illustrate a convenient method of making them if the number of pairs is less than 100. The related pairs are listed in the first and second columns of the table, under A and B. The relative positions or ranks are listed in the third and fourth columns, under R.P._A and R.P._B. The gains in relative position of the B's over the A's—that is, the positive differences, $\text{R.P.}_B - \text{R.P.}_A$—are listed, together with the negative differences, in the fifth column, under G. The squares of the differences between ranks are listed in the sixth column, under D^2. The deviations from the approximate C.T.'s (39.5 and 51) are listed in the seventh and eighth columns under x and y The $x \cdot y$ products are listed in the ninth and tenth columns, the + values in the ninth and the — values in the tenth, under $+xy$ and $-xy$. The x^2s and y^2s are listed in the eleventh and twelfth columns. That is, the table headings have meanings as shown below.

A	Measure in A.
B	Related measure in B.
$R.P._A$	Rank (*i. e.*, eelative position) in A.
$R.P._B$	Rank (*i. e.*, relative position) in B.
G	$R.P._B - R.P._A$.
D^2	$(R.P._B - R.P._A)^2$.
x	Deviation from 39.5 in A (in half-steps).
y	Deviation from 51 in B (in steps).
$+xy$	Products of like-signed pairs.
$-xy$	Products of unlike-signed pairs.
x^2	
y^2	

The calculation of r by the Spearman "footrule for correlation," wherein

$$R = 1 - \frac{6\Sigma G}{n^2 - 1},$$

uses only column 5. It is shown in I. of Table 41. The calculation of r by the squared differences in relative position uses only column 6. It is shown in II. of Table 41. The calculation of r by the percentage of unlike signed pairs uses only the signs of columns 7 and 8. It is shown in III. of Table 41. The calculation of r by the Pearson method uses columns 7, 8, 9 and 10. It is shown in IV. of Table 41.[6] The calculation of r as the median of the x/y and y/x ratios, with allowance for the variability of A and the variability of B, is shown in V. of Table 41.

If the number of related pairs is over 100, the use of relative positions is inadvisable, and the computations for U, $\Sigma x \cdot y$, Σx^2, Σy^2, mid x/y, mid y/x, v_1 and v_2 are best made after the data have been arranged after the general plan shown in Table 42. In Table 42, which is for the facts given in Table 27 (on page 146), when treated as divergences from 20 (for *a-t* words marked) and 55 (for *A*'s marked), each pair is represented by a line placed *under* the appropriate step of the "*a-t words*" scale and *opposite* the appropriate

---

[6] In practise, columns 11 and 12 of Table 40 would not be filled out as is shown here. The labor of adding would be much economized, as by replacing the two 81's by 162, the four 49's by 196, the five 25's by 125, etc.

TABLE 40

1 A	2 B	3 R.P.$_A$	4 R.P.$_B$	5 G	6 D^2	7 x	8 y	9 +xy	10 −xy	11 x^2	12 y^2
31	36	1	3	+2	4	−17	−15	255		289	225
33	39	2	5.5	+3.5	12.25	−13	−12	156		169	144
34	46	3	15	+12	144	−11	−5	55		121	25
35	32	4.5	1	−3.5	12.25	−9	−19	171		81	361
35	41	4.5	9	+4.5	20.25	−9	−10	90		81	100
36	43	7.5	11	+3.5	12.25	−7	−8	56		49	64
36	35	7.5	2	−5.5	30.25	−7	−16	112		49	256
36	37	7.5	4	−3.5	12.25	−7	−14	98		49	196
36	47	7.5	16.5	+9	81	−7	−4	28		49	16
37	49	12	22	+10	100	−5	−2	10		25	4
37	40	12	7.5	−4.5	20.25	−5	−11	55		25	121
37	51	12	26	+14	196	−5	−0	0		25	0
37	50	12	24.5	+12.5	156.25	−5	−1	5		25	1
37	39	12	5.5	−6.5	42.25	−5	−12	60		25	144
38	44	17	12	−5	25	−3	−7	21		9	49
38	48	17	19	+2	4	−3	−3	9		9	9
38	45	17	13.5	−3.5	12.25	−3	−6	18		9	36
38	52	17	28	+11	121	−3	+1		3	9	1
38	52	17	28	+11	121	−3	+1		3	9	1
39	40	22.5	7.5	−15	225	−1	−11	11		1	121
39	45	22.5	13.5	−9	81	−1	−6	6		1	36
39	53	22.5	30.5	+8	64	−1	+2		2	1	4
39	42	22.5	10	−12.5	156.25	−1	−9	9		1	81
39	57	22.5	37	+14.5	210.25	−1	+6		6	1	36
39	49	22.5	22	−.5	.25	−1	−2	2		1	4
40	47	28.5	16.5	−12	144	+1	−4		4	1	16
40	50	28.5	24.5	−4	16	+1	−1		1	1	1
40	55	28.5	34.5	+6	36	+1	+4	4		1	16
40	49	28.5	22	−6.5	42.25	+1	−2		1	1	4
40	59	28.5	39	+10.5	110.25	+1	+8	8		1	64
40	48	28.5	19	−9.5	90.25	+1	−3		3	1	9
41	48	34.5	19	−15.5	240.25	+3	−3		3	9	9
41	56	34.5	36	+1.5	2.25	+3	+5	15		9	25
41	53	34.5	30.5	−4	16	+3	+2	6		9	4
41	60	34.5	34.5	0	0	+3	+9	27		9	81
41	58	34.5	38	+3.5	12.25	+3	+7	21		9	49
41	55	34.5	34.5	0	0	+3	+4	12		9	16
42	66	40	47.5	+7.5	56.25	+5	+15	75		25	225
42	54	40	32.5	−7.5	56.25	+5	+3	15		25	9
42	52	40	28	−12	144	+5	+1	5		25	1
42	66	40	47.5	+7.5	56.25	+5	+15	75		25	225
42	61	40	42.5	+2.5	6.25	+5	+10	50		25	100
43	61	44	42.5	−1.5	2.25	+7	+10	70		49	100
43	54	44	32.5	−11.5	132.25	+7	+3	21		49	9
43	73	44	50	+6	36	+7	+22	154		49	484
44	63	46.5	45	−1.5	2.25	+9	+12	108		81	144
44	70	46.5	49	+2.5	6.25	+9	+19	171		81	361
45	62	48.5	44	−4.5	20.25	+11	+11	121		121	121
45	60	48.5	40.5	−8	64	+11	+9	99		121	81
47	65	50	46	−4	16	+15	+14	210		225	196

TABLE 41

Calculations of r from the Data of Table 40, by Various Methods

I. $\Sigma g = 165$ or 171, according to the direction chosen. Use 168.
$R = 1 - 6(168)/(2500 - 1)$ $R = .5966$
Using $r = \sin \frac{\pi}{2} R$ (*i. e.*, Table 37), $r = .806$.
Using $r = 2 \cos \frac{\pi}{3} (1 + R) - 1$ (*i. e.*, Table 38), $r = .824$.

II. $\Sigma D^2 = 3171$ $\rho = 1 - \frac{6(3171)}{50(2499)}$ $\rho = .848$
Assuming form A for the distributions, and so using Table 36, $r = .859$.

III. There are 9 U-pairs, 1 zero-pair, and 40 L-pairs.
By Table 39, $r = .832$.

IV. $\Sigma xy = 2468$ $\Sigma x^2 = 2074$ $\Sigma y^2 = 4385$ $r = \frac{2486}{\sqrt{2074}\sqrt{4385}}$, or $= .818$

V. Listing the x/y ratios, roughly in order of magnitude, we have:
9 negative ratios
9 ratios from 0 to $\frac{1}{3}$
$\frac{5}{12}$ $\frac{7}{16}$ $\frac{3}{7}$ $\frac{3}{7}$
$\frac{5}{11}$ $\frac{9}{19}$ $\frac{9}{19}$
$\frac{7}{14}$ $\frac{3}{6}$ $\frac{1}{2}$ $\frac{5}{10}$
$\frac{4}{5}$ $\frac{7}{10}$ $\frac{3}{4}$ $\frac{9}{12}$
$\frac{7}{8}$
$\frac{9}{10}$
15 ratios of 1 or over

Listing the y/x ratios, roughly in order of magnitude, we have:
9 negative ratios
9 ratios from 0 to $\frac{3}{4}$
$\frac{14}{17}$ $\frac{11}{13}$ $\frac{9}{11}$ $\frac{14}{15}$
1 1
$\frac{10}{9}$ $\frac{8}{7}$
24 ratios of 1.25 or over

The mid x/y is between $\frac{9}{19}$ and $\frac{1}{2}$.
The mid x/y is .487.

The mid y/x is between $\frac{10}{9}$ and $\frac{8}{7}$.
The mid y/x is 1.127.

$\sigma_A = \sqrt{\frac{2074}{50}}$, and $\sigma_B = \sqrt{\frac{4385}{50}}$. Hence, $v_1 = 1.45$ and $v_2 = .69$.

(Mid $x/y)v_1 = .706$.

(Mid $y/x)v_2 = .778$.

$r = \sqrt{.706 \times .778}$, or .741; or, approximately, $r = \frac{.706 + .778}{2}$, or .741.

step of the "*A*'s *marked*" scale. The average of the y values related to each value of x is given under Av. y; the sum of the $x \cdot y$ products in the case of each array of y's is got by multiplying the Av. y in question by x and then multiplying the product, so obtained, by the number of cases in that array; the $\Sigma x \cdot y$ for the entire series is got by adding these smaller product-sums, which are recorded under $f(xy)$; the calculation of Σx^2 and Σy^2 is abbreviated by grouping as shown under $f(x^2)$ and $f(y^2)$; the other facts and arrangements of Table 42 are self-explanatory.

In general the following procedure is advisable in measuring any correlation by the amount methods:

TABLE 42

	x	f	Av. y	$f(xy)$		$f(x^2)$
10	−10	2	−11.50	230		200
11	− 9	3	− 6.33	171		243
12	− 8	2	− 9.50	152		128
13	− 7	2	− 1.50	21		98
14	− 6	2	− 2.00	24		72
15	− 5	9	− 5.56	250		225
16	− 4	7	− .57	16		112
17	− 3	7	− 8.86	186		63
18	− 2	12	− 2.33	56		48
19	− 1	4	+ 5.50		−22	4
20	0	18	0	0		0
21	+ 1	14	− .93		−13	14
22	+ 2	8	+ 3.88	62		32
23	+ 3	6	+ 3.00	54		54
24	+ 4	4	+ 6.75	108		64
25	+ 5	6	+10.17	305		150
26	+ 6	2	+ 3.50	42		72
27	+ 7	6	+13.17	553		294
28	+ 8	3	+ 6.33	152		192
29	+ 9	1	+14.00	126		81
30	+10	3	+ 8.00	240		300
31	+11					
32	+12					
33	+13					
34	+14	1	+18.00	252		196
	$n =$	122	$\Sigma xy =$	2965	$\Sigma x^2 =$	2642

	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
36	|																								
37						|																			
38						|					|													
39								|																	
75																									
76											|														
77													|												
78															|	|		|							
79																									
80																									
81																|					|			

y	f	$f(y^2)$
19	1	361
18	1	324
17	2	578
16	1	256
20		
21	1	441
22	1	484
23	3	1587
24		
25		
26	2	1352

$\Sigma y^2 = 12690$

1. Keep the measures in as fine a grouping as they originally had.
2. Tabulate them in a single-entry correlation table, letting one little line represent one pair; the location of this line, beneath the scale for *A* and opposite the scale for *B*, representing the two measures.
3. Choose, as an approximate point of reference for the *A*'s, that midpoint of a step on the *A*-scale or that point just between two steps on the *A*-scale which is nearest the desired point of reference. Do likewise for the *B*'s. What the desired point of reference will be in any case depends upon what theoretical or practical question the calculation of the *r* is to answer.
4. Enter, in a row at the bottom, the frequencies of each step of the *A*-scale. Enter, in a column at the right, the frequencies of each step of the *B*-scale.
5. Restate the *A*-scale as an *x*-scale, in divergences from the approximate point of reference in units of a step or half-step, according as the approximate point of reference is at the middle of a step or just between two steps. Do likewise for the *B*-scale, turning it into a *y*-scale.
6. Calculate the central tendency of each column under each step of the scale for the *A*'s, and do not fail, whatever single value may be later calculated to represent the general drift of the relation, to give with it this list of the central tendencies of the *B*'s related to the respective different values of *A*.

§ 40. *The Correction of Correlation-Coefficients for the 'Attenuation' Due to Chance Inaccuracies in the Original Paired Measures*

The discussion of measurements of relations so far presupposes that the facts related are measured exactly. There will, however, in mental and social measurements commonly be a considerable error in each individual fact of those to be related. For instance, in the illustration used in Table 42 the "*A*'s marked by an individual" is a score depending upon only one trial of 60 seconds. With many trials on many different occasions, the individuals concerned would attain somewhat different measures. So also with the "*a-t* words marked." Let us call $r_{\text{acc. m.}}$ the *r* which would be obtained from accurate measures in both facts for all of the related pairs; and let us call $r_{\text{app. m.}}$ the *r* which is in fact calculated from the single measures. If there is, in reality, any correlation, direct or inverse, $r_{\text{acc. m.}}$ will be farther from 0 than $r_{\text{app. m.}}$; for the influence of chance inaccuracy in the measures to be related *is always to produce zero correlation.* If two series of pairs of values are due entirely to chance the correlation will be zero, and in so far as they are at all due to chance, the correlation will be reduced toward zero.

The chance variation, which in the long run cuts its own throat in the case of averages, can not, in the case of a correlation, be thus rendered innocuous by mere numbers. For instance, the true correlation between the volume of bodies of water at constant pressure and temperature, etc., and their weight is + 1.00. Suppose now that the true measures for ten pairs were:

Case	Vol.	Wt.
A	2	4
B	4	8
C	6	12
D	7	14
E	8	16
F	9	18
G	10	20
H	11	22
I	13	26
J	15	30

The correlation is evidently + 1.00.

Suppose the person measuring them got, instead of these figures, certain chance variations from them due to the error of his measuring.

If the reader will distribute by chance, among these 20 measures, 20 errors, say 5 of + 2, 5 of − 2, 4 of + 3, 4 of − 3, 1 of + 4 and 1 of − 4, and will then calculate again the coefficient, he will find it to be less than before. It he will let the chance errors be larger *e. g.*, 5 each of + 4 and − 4, 4 each of + 6 and − 6 and 1 each of + 10 and − 10, the coefficient will be still more reduced. The same will hold regardless of whether 10 or 10,000 pairs of related values are taken.

To correct for this "attenuation" of the coefficient by chance errors in the data, it is necessary to have at least two independent measures of the measures to be related. When these are at hand the procedure is as follows:

Let A and B be the facts to be related.

Let p be a series of exact measures of A.

Let q be the related series of exact measures of B.

Let r_{pq} be the coefficient of correlation of A and B, obtainable from the two series p and q. r_{pq} is thus the required real correlation.

Let p_1 and p_2 be two independent series of measures of A.

Let q_1 and q_2 be two independent series of measures of B.

Let $r_{p_1q_1}$ be the correlation when the first measure of A and the first measure of B are used.

Let $r_{p_1q_2}$ be the correlation when the first measure of A and the second measure of B are used.

Let $r_{p_2q_1}$ be the correlation when the second measure of A and the first measure of B are used.

Let $r_{p_2q_2}$ be the correlation when the second measure of A and the second measure of B are used.

Let $r_{p_1p_2}$ be the correlation between the two measures of A.

Let $r_{q_1q_2}$ be the correlation between the two measures of B.

It is understood that the pairing is the same in every case. Then

$$r_{pq} = \frac{\sqrt[4]{(r_{p_1q_1})(r_{p_1q_2})(r_{p_2q_1})(r_{p_2q_2})}}{\sqrt{(r_{p_1p_2})(r_{q_1q_2})}}.$$

Labor can be economized, and a very probably better correction obtained, by using

$$r_{pq} = \frac{\sqrt{(r_{p_1q_2})(r_{p_2q_1})}}{\sqrt{(r_{p_1p_2})(r_{q_1q_2})}}.$$

A second method[7] of allowing for the inaccuracy of the original measures of the facts to be related is based upon the fact that an increase in the number of measures of each of such facts increases its accuracy. From the increase in the closeness of the relationship as we use the central tendency of 2, 3, 4, 5 . . . trials of each individual, we may prophesy what the relationship would be, if we had at hand measures from so many trials of all the individuals as to give the status of each one exactly.

[7] For a further description of this method and the first method as well see the article in the *Am. J. of Psy.*, for January, 1904, by C. Spearman, to whom the formulæ are due.

Let r_{pq} be the coefficient of correlation that would be found if the measures of the related facts, A and B, were perfectly exact.

Let m be the number of independent measures of A, $p_1p_2p_3$, etc.

Let n be the number of independent measures of B, $q_1q_2q_3$, etc.

Let $r_{p'q'}$ be the average of the correlations between each series of values obtained for trait A, with each series obtained for trait B.

Let $r_{p''q''}$ be the correlation obtained when $p_1p_2p_3$, etc., are combined to give the measure of trait A, when, that is, each individual is represented by his most likely central tendency in trait A, and when $q_1q_2q_3$ are similarly combined to give the measure of trait B. Then

$$r_{pq} = \frac{\sqrt[4]{mn}(r_{p''q''}) - r_{p'q'}}{\sqrt[4]{mn} - 1}.$$

This second formula has not been accepted as necessarily valid and should be used only provisionally, until it has been verified by theory or experiment, but it is obvious that some empirical formula of the sort could be found to give the expected r from absolutely accurate measures on the basis of the change in the r as the measures approach nearer such absolute accuracy. The first formula is valid in so far as the difference between any two of the original measures of the same fact is due to truly random sampling.

Useful as these formulæ for correction of attenuation due to inaccurate measures are, it is wise not to overwork them by substituting their use for the attainment of reasonably precise original measures. The beginner, at all events, may best secure, in the case of correlations, original measures, the $\sigma_{\text{true-obtained}}$[8] of which is not over 5 per cent. of their amount.

§ 41. *Estimating the Correlation that Would Be Found if the Original Paired Measures Could Be Freed from the Effects of Irrelevant Factors*

It is obvious that, in order to measure the essential correlation between fact A and fact B, we should have a series of pairs of amounts related only through the relationship of A to B. But unless great care is taken in the selection of the data, other factors affecting the relationship of the amounts are sure to enter. Thus in relating mental capacities, if we use children of different ages, the factor of age, as well as the intrinsic relationship between the traits, is at work. The real correlation between a city's lighting and its expense for police protection might be inverse, but actual correlations of the per capita expense for the two items in American cities might show a direct relationship due to the entrance of the factor, municipal expensiveness as a whole. The influence of heredity can not be inferred from fraternal correlation until a discount is made for the factor, similar training.

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[8] See the next chapter for the explanation of this term.

Spearman has suggested the terms—Constriction, Dilation and Distortion—for the effects of the improper admission or exclusion of factors. I quote his description[9] and corrective formulæ.

"Now, all such elements in a correlation as are foreign to the investigator's explicit or implicit purpose will, like the attenuating errors, constitute impurities in it and will quantitatively falsify its apparent amount. This will chiefly happen in two ways.

"4. 'Constriction' and 'Dilation.'

" Any correlation of either of the considered characteristics will have been admitted irrelevantly, if it has supervened irrespectively of the original definition of the correspondence to be investigated. The variations are thereby illegitimately constrained to follow some irrelevant direction so that (as in the case of Attenuation) they no longer possess full amplitude of possible correlation in the investigated direction; the maximun instead of being 1 will be only a fraction, and all the lesser degrees of correspondence will be similarly affected; such a falsification may be called 'constriction.' Much more rarely, the converse or 'dilation' will occur, by correlations being irrelevantly excluded. The disturbance is measurable by the following relation:

$$r_{pq} = \frac{r_{pq}'}{\sqrt{1 - r_{pv}^2}},$$

where r_{pq}' = the apparent correlation of p and q, the two variables to be compared.

r_{pv} = the correlation of one of the above variables with a third and irrelevantly admitted variable v,

and r_{pq} = the real correlation between p and q, after compensating for the illegitimate influence of v.

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[9] *Am. J. of Psy.*, 1904, Vol. 15, pp. 94–96, *passim.*

"Should any further irrelevant correlation, say r_{pw}, be admitted, then

$$r_{pq} = \frac{r_{pq}'}{\sqrt{1 - r_{pv}^2 - r_{pw}^2}}.$$

In the reverse case of 'dilation,'

$$r_{pq} = r_{pq}' \cdot \sqrt{1 - r_{pv}^2 - r_{pw}^2} \cdots .$$

"Distortion occurs whenever the two series to be compared together both correspond to any appreciable degree with the same third irrelevant variant. In this case, the relation is given by

$$r_{pq} = \frac{r_{pq}' - (r_{pv})(r_{qv})}{\sqrt{(1 - r_{pv}^2)(1 - r_{qv}^2)}},$$

where r_{pq}' = the apparent correlation between p and q, the two characteristics to be compared,
r_{pv} and r_{qv} = the correlation of p and q with some third and perturbing variable v,
and r_{pq} = the required real correlation between p and q, after compensating for the illegitimate influence of v."

Should the common correspondence with v have been irrelevantly excluded instead of admitted, the relation becomes

$$r_{pq} = r_{pq}' \cdot \sqrt{(1 - r_{pv}^2)(1 - r_{qv}^2)} + (r_{pv})(r_{qv})$$

§ 42. *The Dependence of the Meaning of a Coefficient of Correlation upon the Values that Are Paired*

The facts to be correlated in the mental and social sciences may be: (1) the varying conditions of a trait in an individual (to be correlated with corresponding conditions in him of some other trait), or (2) the varying conditions of a trait found in different individuals of a group (to be correlated with the conditions found in some other trait in the same individuals), or (3) the varying central tendencies of a trait found in different subgroups of a larger group or collection of groups (to be correlated with the central tendencies found in the case of some other trait in the same subgroups), or many other series of pairs.

For example, one may seek (Case 1) the correlation between the quickness of perception of an individual at various times and his quickness of movement at corresponding times. Or one may seek (Case 2) the correlation between the quickness of perception in general of Jones, Smith, Brown, etc., and the quickness of movement possessed in general by the same individuals. Or (Case 3) one may seek the correlation between the general quickness in perception of races and their quickness of movement.

It should be noted that the differences in the three cases have nothing to do with the mere number of individuals studied. The essential differences would remain if we used a million cases to determine the correlation of two traits within an individual, only a hundred thousand to determine the correlation among individuals and only ten thousand to determine it for races. The essential difference is in the questions to be solved. From them it follows also that in Case 1, if several individuals are studied, a number of pairs of measures for each individual will be used and the coefficient of correlation in each individual will be worked out separately. If the results from different individuals are then combined they will be combined as a group of facts according to the methods of Chapter III. In Case 2, on the contrary, a single pair of measures will represent the correlation in any one individual and these pairs will be combined according to the method of the present chapter. In Case 3 a single pair of figures will represent the correlation in each subgroup.

The problem of measurement itself is the same for three cases, the difference being in the data used and the consequent meaning of the coefficient of correlation obtained. To any one of the following series of related pairs the mode of procedure discussed in this chapter is applicable.

Related by Identity of Conditions
Trait T and trait T_1 in individual A under conditions C_1
Trait T and trait T_1 in individual A under conditions C_2
Trait T and trait T_1 in individual A under conditions C_3

Related by Identity of the Individual
Trait T and trait T_1 in group, ten-year-olds, in individual I_1
Trait T and trait T_1 in group, ten-year-olds, in individual I_2
Trait T and trait T_1 in group, ten-year-olds, in individual I_3

Related by Identity of the Sub-group
Trait T and trait T_1 in group, all men, in sub-group Chinese.
Trait T and trait T_1 in group, all men, in sub-group Negroes.
Trait T and trait T_1 in group, all men, in sub-group Indians.

It is perhaps needless to point out that the existence of a certain relation within an individual does not imply anything about the relation within a group of individuals, nor that again about the relation within a group of groups. Individuals may be happier when they are richer, but rich individuals amongst Americans may be no happier than poor individuals, and from neither fact could we infer that the American population would be happier or less happy than the Chinese or the Negro population.

For similar reasons the nature and amount of a correlation will depend upon the group selected. If, for instance, the correlation between knowledge of history and knowledge of English literature is measured in the group, high-school graduates, by using the deviations of individuals from the high-school graduates' averages in the two traits, the correlation will be less close than if we use the group, all people. The correlation between height and weight will be less close if measured in the group, 18-year-olds, than if measured in all children under twenty. Any relation so calculated should always be thought of as the correlation of deviations from the assigned points of reference in the two traits in the case of the individuals of *the group in question.* To assume that the correlation found in any given group holds good also for a different group is valid only if the given group is a random selection from the other group.

PROBLEMS

41. Arrange a correlation-table, pairing the two series given below so that r when calculated will be approximately .8.

42. Arrange a second correlation table, pairing them so that r will be approximately .5.

43. Pair them so that r will be approximately .2.

Series I.: $-10, -8, -7, -6, -6, -5, -5, -4, -4, -4, -3, -3, -3, -2, -2, -2, -2, -1, -1, -1, -1, +1, +1, +1, +1, +2, +2, +2, +2, +3, +3, +3, +3, +4, +4, +5, +5, +6, +6, +7, +8, +10$.

Series II.: $-5, -4, -4, -3, -3, -3, -2, -2, -2, -2, -2, -2, -1, -1, -1, -1, -1, -1, -1, 0, 0, 0, 0, +1, +1, +1, +1, +1, +1, +1, +2, +2, +2, +2, +2, +3, +3, +3, +3, +4, +4, +5$.

44. Compute r by all possible methods for the following series of pairs, using 115 and 80 as C.T.'s.

Individual	Score in *A*	Score in *B*
a	106	60
b	109	70
c	111	77
d	112	65
e	113	82
f	114	81
g	114	79
h	115	80
i	115	73
j	115	75
k	116	78
l	116	95
m	117	87
n	118	85
o	119	83
p	121	100
q	124	90

45. Compute the Pearson coefficients for *A* with *B*, *A* with *C*, and *A* with *D* in Table 31 (page 158).

46. Compute v_1, v_2, the mid x/y ratio, and the mid y/x ratio and r by the formula $r = \sqrt{[(\text{mid } x/y)v_1][(\text{mid } y/x)v_2]}$ for the data of Table 42.

To save time, treat ratios with zero in the denominator as extreme negatives when the numerator is negative, and as extreme positives when the numerator is positive. In practice, the mid-ratio method would not be used without distributing the zero measures. The principle of distribution would be that used for the "percentage of like-signed pairs" method.

CHAPTER XII

THE RELIABILITY OF MEASURES

§ 43. *Dependence upon the Number of Separate Measures of the Fact in Question and upon their Variability*

WHEN from a limited number of measurements of a fact, say of *A*'s monthly expenses or *B*'s ability in perception, we calculate its average, the result is not, except by chance, the true average. For, obviously, one more measurement will, unless it happens to coincide with the average obtained, change it. For instance, the first 30 measures of *H*'s ability in reaction time gave the average .1405; the next seven measures being taken into account, the average became .1400; with the next seven it became .1406 − ; with the next seven, .1406 +. By the true average we mean the average that would come from all possible measures of the fact in question. The actual average obtained from a limited finite number of these measures is, except by chance, only an approximation toward the true average. So also with the accuracy of the measure of variability obtained. The true variability is that manifested in the entire series of measurements of the trait; the actually obtained variability is an approximation toward it. The true average and the true variability of a group mean similarly the measures obtained from a study of all the members of the group.

It is necessary, then, to know how many trials of an individual, how many members of a group, must be measured, to obtain as accurate knowledge as we need. Or, to speak more properly, it is necessary to know how close to the true measure the result obtained from a certain finite number of measures will be.

It is clear that the true average of any set of measures is the average calculated from all of them. If the average we actually obtain is calculated from samples chosen at random, it will probably diverge somewhat from the average calculated from all. So also with obtained and true measures of total distribution, variability, difference and relationship. We measure the unreliability of any obtained measure by its probable divergence from the true measure.

It is clear also that the divergence of any measure due to a limited number of measures from the corresponding measure due to the entire series, will vary according to what particular samples we hit upon, and that if the samples are taken at random this variation in the amount of divergence will follow the laws of probability. For these laws, based on the algebraic law expressing the number of combinations of *r* things taken *n* at a time, will account for the difference between the constitution of a total series and the constitution of any group of things chosen at random from it, consequently for the differences between any two measures due respectively to these two constitutions.

We have, consequently, to find the distribution of a probable *divergence* (of obtained from true or of true from obtained) and know beforehand, in cases of random sampling, that it will have its mode at 0 (since all that we do know about the true is that it is more likely to be the obtained measure than to be any other one measure). What we need to know is its form and variability, to know, that is, how often we may expect a divergence of .01, how often one of .02, how often one of .03, etc. Suppose our obtained measure to be 10.4 and the distribution of the probable divergence of its corresponding true measure from it to be known to be as follows:

−1.1 to − .9	1 or 0.1 per cent.
− .9 to − .7	10 or 1 per cent.
− .7 to − .5	45 or 4.5 per cent.
− .5 to − .3	120 or 12 per cent.
− .3 to − .1	210 or 21 per cent.
− .1 to + .1	252 or 25 per cent.
+ .1 to + .3	210 or 21 per cent.
+ .3 to + .5	120 or 12 per cent.
+ .5 to + .7	45 or 4.5 per cent.
+ .7 to + .9	10 or 1 per cent.
+ .9 to +1.1	1 or 0.1 per cent.

We can say: "The true measure will not rise above 11.3 (10.4 + .9) in more than one case out of 1,024," or, "The chances are over 1,000 to 1 against the measure being over 11.3," or, "The chances are nearly 99 to 1 against the true measure being over 11.1," or, "The chances are about 8 to 1 against the true measure differing from 10.4 either above or below by more than .5."[10]

If the form of the distribution of the divergence of the true from the obtained measure were known, its variability would be the only measure needed. The form will, if it is fairly large, always be fairly near to Form *A* and it is customary to disregard the very slight error involved and assume the form to be that of Form *A*.

The problem of determining the reliability of any measure due to a limited series of samples is, then, to determine the variability of the fact, *divergence of true from obtained measure.*

It is clear that the more nearly the number of samples taken approaches the number of things they represent, the closer the obtained measure will, in general, be to the true measure. In other words, the dispersion of the divergence ($M_{\text{true}}' - M_{\text{obt.}}$) about zero will be less. It is clear also that the less the variability amongst the individual samples, the less will be the probable variability of the divergence of the obtained from the true measure of central tendency. For instance, if men range from 4 to 7 feet in height, averaging 5 feet 8 inches, we can not possibly get an average more than 1 foot 8 inches wrong, while if they range from 2 to 10 feet, we may make an error of 3 feet 8 inches. The same holds true for the divergence of obtained from true variability.

The derivation of the formulæ expressing the variability of the probable divergence of the true from the obtained measure of central tendency or of variability in terms of the number of cases and the variability of the obtained distribution need not concern us.

The formulæ in common use are given in Section 44.

§ 44. *Formulæ for the Variability of the Probable Divergence of a True Measure from Its Corresponding Obtained Measure*

For the unreliability[11] of an average.

$$\sigma_{\text{t. av.—obt. av.}} = \frac{\sigma_{\text{dis.}}}{\sqrt{n}},$$

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[10] It may appear strange to talk about the true measure, which is a fixed value, "rising above" or "being over," but if the reader will bear in mind that we do not know just where it is fixed, but do know the *probability* of its being at this or that point, he will not misunderstand the terms used. They could not well be avoided without much circumlocution.

[11] It is customary to speak of the variability of the divergence of a true from an obtained measure as the measure's *error*. Thus $\sigma_{\text{t. av.—obt. av.}}$ is called the

in which

$\sigma_{\text{t. av.—obt. av.}}$ = the mean square deviation of the probable divergence of the true from the obtained average.[12]

$\sigma_{\text{dis.}}$ = the mean square deviation of the distribution[13] the unreliability of whose average is in question.

n = the number of measures in the distribution.

The use of the formula to predict facts will be seen best in a concrete case. For instance, let $A_{\text{obt.}}$ = the obtained average: let $\sigma_{\text{dis.}}$ = the variability (mean square or standard deviation) of the distribution: let $A_{\text{t.}}$ = the true average that would be obtained from an infinite number of measures. Suppose $A_{\text{obt.}}$ is 20.2, that $\sigma_{\text{dis.}}$ is 4.2, and that n is 300. Then the probable condition of the divergence, $A_{\text{t.}} - A_{\text{obt.}}$, is to have a mode at zero and a variability ($\sigma_{\text{t.—obt.} A}$) about that mode of $4.2/\sqrt{300}$, or .242. The distribution of the probable divergence of the true from the obtained average is then as shown in Fig. 91, which is constructed from the data: Form of distribution is Form A; mode = zero; σ = .242.

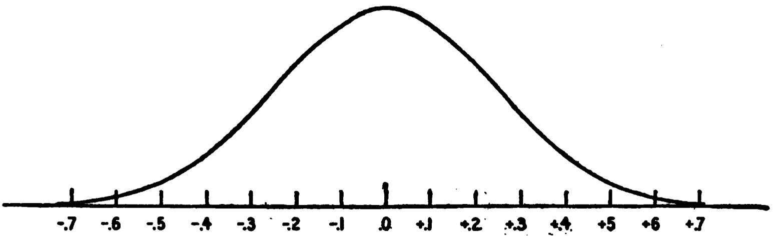

FIG. 91.

mean square error of the obtained average; P.E.$_{\text{t. r.—obt. r.}}$ is called the *probable error* of the obtained coefficient of correlation; A.D.$_{\text{t. diff.—obt. diff.}}$ is called the *average error* of the obtained difference. These terms are somewhat ill chosen, as there is really no "error," but only a varying degree of probable approximation. I shall, therefore, use the word *unreliability* throughout.

[12] The meaning of the "probable divergence" of the true measure ($M_{\text{t.}}$) from that obtained from a series of, say, fifty measures ($M_{\text{obt.}}$) is the *actual* divergences from $M_{\text{t.}}$ that would be found using all the $M_{\text{obt.}}$'s of an infinite number of similar series of fifty measures. That is, if $\sigma_{\text{t. m.—obt. m.}} = 1.5$, it means that if a hundred thousand or so such experiments were to be made, each giving fifty measures, and if the divergences of the hundred thousand $M_{\text{obt.}}$'s from $M_{\text{t.}}$ were computed, they would be found to be distributed around zero as a central tendency, with a mean square deviation of 1.5.

[13] Since, to measure unreliability, we have to measure the variability of a divergence and shall need to use terms similar to those used in measuring the variability of things or conditions, it will be well to name the average deviation of a distribution of a thing or condition A.D.$_{\text{dis.}}$. Similarly, σ and P.E. in the sense hitherto used will now be called $\sigma_{\text{dis.}}$ and P.E.$_{\text{dis.}}$.

From it one can, by proper means,[14] show that the probable divergence of $A_{t.}$ from 20.2 (the value of $A_{obt.}$) ranges between $-$ .726 and $+$ .726 in about 997 cases out of 1,000, between $-$ .242 and $+$ .242 in 683 cases out of 1,000, between $-$ .40 and $+$ .40 in 900 cases out of 1,000. In other words, the chances are 997 to 3, or 332 to 1, that the true average will not deviate from the obtained by more than .726; 683 to 317, or over 2 to 1, against a deviation of over .242; and 900 to 100, or 9 to 1, against a deviation of over .40. In still different words, the chances are 2 to 1 that the true average lies between 19.958 and 20.442; 9 to 1 that the true average lies between 19.8 and 20.6; 332 to 1 that the true average lies between 19.474 and 20.926.

The Unreliability of a Median.—The above formulæ, multiplied by 1.25331 (or roughly 1¼), may be used for the variability of the divergence of the obtained from the true median. That is,

$$\sigma_{\text{t. med.—obt. med.}} = \frac{5}{4} \cdot \frac{\sigma_{\text{dis.}}}{\sqrt{n}}$$

The Unreliability of a σ and of a Q.—The gross unreliability is less for the variability than for the central tendency of the same fact, $\sqrt{2n}$ replacing $\sqrt{n}$ in the denominator. The formula for the mean square deviation of the divergence of true from obtained σ is:

$$\sigma_{\text{t. }\sigma\text{—obt. }\sigma} = \frac{\sigma_{\text{dis.}}}{\sqrt{2n}}$$

For the unreliability of a Q we may use

$$\sigma_{\text{t. }Q\text{—obt. }Q} = \frac{1.11\sigma_{\text{dis.}}}{\sqrt{2n}}$$

The Unreliability of a Difference.—The unreliability of a measure of a difference $(A - B)$ is measured by the probable divergence of the true difference from the obtained difference. *True difference* may mean either one of two things—either the difference that would be the central tendency of an infinite number of measures of the difference, or the difference that would be found between the true measure of A and the true measure of B.

In the first meaning the unreliability of the difference D is the unreliability of the obtained central tendency of the measures—$d_1, d_2 \ldots d_n$—each a sample of a certain sort of difference—and is to be estimated just as is the unreliability of any central tendency (see p. 188ff).

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[14] These means will be made clear in Chapter XIII.

In the second meaning, the unreliability is of $A_{\text{obt.}} - B_{\text{obt.}}$—the obtained difference between a measure A and a measure B—and means the divergence of the probable $A_{\text{true}} - B_{\text{true}}$ from $A_{\text{obt.}} - B_{\text{obt}}$. Consider first the distribution of the probable $A_{\text{t.}} - B_{\text{t}}$.

The probable true measure A_{t} is distributed about $A_{\text{obt.}}$ as a mode and the probable true measure $B_{\text{t.}}$ is distributed about $B_{\text{obt.}}$ as its mode. The probable true difference—that is, $A_{\text{t.}} - B_{\text{t.}}$—is a variable with its mode at $A_{\text{obt.}} - B_{\text{obt.}}$ and with decreasing frequencies as we take $A_{\text{obt.}} - B_{\text{obt.}} + 1$, $A_{\text{obt.}} - B_{\text{obt.}} + 2$, etc., or $A_{\text{obt.}} - B_{\text{obt.}} - .1$, $A_{\text{obt.}} - B_{\text{obt.}} - 2$, etc. This may be seen most clearly in a concrete case such as follows:

Suppose that $A_{\text{obt.}} = 50$, and that $B_{\text{obt.}} = 42$; that the probable divergence of $A_{\text{t.}}$ from $A_{\text{obt.}}$ is as given below under I.; and that the probable divergence of $B_{\text{t.}}$ from $B_{\text{obt.}}$ is as given below under II.

	Frequency	
Divergence	I	II
−2 to −3	1	1
−1 to −2	5	5
0 to −1	10	10
0 to +1	10	10
+1 to +2	5	5
+2 to +3	1	1

To find the probable $A_{\text{t.}} - B_{\text{t.}}$. From I. and II. we get, as probable values of A_{true} and B_{true}, III. and IV.

III		IV	
	A True		B True
48 to 47	1	40 to 39	1
49 to 48	5	41 to 40	5
50 to 49	10	42 to 41	10
50 to 51	10	42 to 43	10
51 to 52	5	43 to 44	5
52 to 53	1	44 to 45	1

Using for each distance its midpoint value, the probable A_{true} is as in V., and the probable B_{true} is as in VI.:

V		VI	
A True		*B* True	
47.5	1	39.5	1
48.5	5	40.5	5
49.5	10	41.5	10
50.5	10	42.5	10
51.5	5	43.5	5
52.5	1	44.5	1

From these probable values of A_{true} and B_{true} we get the following probable values of $A_{\mathrm{t.}} - B_{\mathrm{t.}}$:

One 39.5 of $B_{\mathrm{t.}}$ with one 47.5 of $A_{\mathrm{t.}}$ gives 1 difference of 8
One 39.5 of $B_{\mathrm{t.}}$ with five 48.5s of $A_{\mathrm{t.}}$ gives 5 differences of 9
One 39.5 of $B_{\mathrm{t.}}$ with ten 49.5s of $A_{\mathrm{t.}}$ gives 10 differences of 10
One 39.5 of $B_{\mathrm{t.}}$ with ten 50.5s of $A_{\mathrm{t.}}$ gives 10 differences of 11
One 39.5 of $B_{\mathrm{t.}}$ with five 51.5s of $A_{\mathrm{t.}}$ gives 5 differences of 12
One 39.5 of $B_{\mathrm{t.}}$ with one 52.5 of $A_{\mathrm{t.}}$ gives 1 difference of 13

Five 40.5s of $B_{\mathrm{t.}}$ with one 47.5 of $A_{\mathrm{t.}}$ gives 5 differences of 7
Five 40.5s of $B_{\mathrm{t.}}$ with five 48.5s of $A_{\mathrm{t.}}$ gives 25 differences of 8
Five 40.5s of $B_{\mathrm{t.}}$ with ten 49.5s of $A_{\mathrm{t.}}$ gives 50 differences of 9
Five 40.5s of $B_{\mathrm{t.}}$ with ten 50.5s of $A_{\mathrm{t.}}$ gives 50 differences of 10
Five 40.5s of $B_{\mathrm{t.}}$ with five 51.5s of $A_{\mathrm{t.}}$ gives 25 differences of 11
Five 40.5s of $B_{\mathrm{t.}}$ with one 52.5 of $A_{\mathrm{t.}}$ gives 5 differences of 12

Ten 41.5s of $B_{\mathrm{t.}}$ with one 47.5 of $A_{\mathrm{t.}}$ gives 10 differences of 6
Ten 41.5s of $B_{\mathrm{t.}}$ with five 48.5s of $A_{\mathrm{t.}}$ gives 50 differences of 7
Ten 41.5s of $B_{\mathrm{t.}}$ with ten 49.5s of $A_{\mathrm{t.}}$ gives 100 differences of 8
Ten 41.5s of $B_{\mathrm{t.}}$ with ten 50.5s of $A_{\mathrm{t.}}$ gives 100 differences of 9
Ten 41.5s of $B_{\mathrm{t.}}$ with five 51.5s of $A_{\mathrm{t.}}$ gives 50 differences of 10
Ten 41.5s of $B_{\mathrm{t.}}$ with one 52.5 of $A_{\mathrm{t.}}$ gives 10 differences of 12

Ten 42.5s of $B_{\mathrm{t.}}$ with one 47.5 of $A_{\mathrm{t.}}$ gives 10 differences of 5
Ten 42.5s of $B_{\mathrm{t.}}$ with five 48.5s of $A_{\mathrm{t.}}$ gives 50 differences of 6
Ten 42.5s of $B_{\mathrm{t.}}$ with ten 49.5s of $A_{\mathrm{t.}}$ gives 100 differences of 7
Ten 42.5s of $B_{\mathrm{t.}}$ with ten 50.5s of $A_{\mathrm{t.}}$ gives 100 differences of 8
Ten 42.5s of $B_{\mathrm{t.}}$ with five 51.5s of $A_{\mathrm{t.}}$ gives 50 differences of 9
Ten 42.5s of $B_{\mathrm{t.}}$ with one 52.5 of $A_{\mathrm{t.}}$ gives 10 differences of 10

Five 43.5s of $B_{\mathrm{t.}}$ with one 47.5 of $A_{\mathrm{t.}}$ gives 5 differences of 4
Five 43.5s of $B_{\mathrm{t.}}$ with five 48.5s of $A_{\mathrm{t.}}$ gives 25 differences of 5
Five 43.5s of $B_{\mathrm{t.}}$ with ten 49.5s of $A_{\mathrm{t.}}$ gives 50 differences of 6
Five 43.5s of $B_{\mathrm{t.}}$ with ten 50.5s of $A_{\mathrm{t.}}$ gives 50 differences of 7
Five 43.5s of $B_{\mathrm{t.}}$ with five 51.5s of $A_{\mathrm{t.}}$ gives 25 differences of 8
Five 43.5s of $B_{\mathrm{t.}}$ with one 52.5 of $A_{\mathrm{t.}}$ gives 5 differences of 9

One 44.5 of $B_{\mathrm{t.}}$ with one 47.5 of $A_{\mathrm{t.}}$ gives 1 difference of 3
One 44.5 of $B_{\mathrm{t.}}$ with five 48.5s of $A_{\mathrm{t.}}$ gives 5 differences of 4
One 44.5 of $B_{\mathrm{t.}}$ with ten 49.5s of $A_{\mathrm{t.}}$ gives 10 differences of 5
One 44.5 of $B_{\mathrm{t.}}$ with ten 50.5s of $A_{\mathrm{t.}}$ gives 10 differences of 6
One 44.5 of $B_{\mathrm{t.}}$ with five 51.5s of $A_{\mathrm{t.}}$ gives 5 differences of 7
One 44.5 of $B_{\mathrm{t.}}$ with one 52.5 of $A_{\mathrm{t.}}$ gives 1 difference of 8

These, being distributed, give the facts of Table 43.

TABLE 43

DISTRIBUTION OF THE PROBABLE $A_{t.} - B_{t.}$

Quantity: Probable $A_{t.} - B_{t.}$	Frequency: Chances out of 1,024
3	1
4	10
5	45
6	120
7	210
8	252
9	210
10	120
11	45
12	10
13	1

This gives the mode of the probable $A_{t.} - B_{t.}$ as 8, which equals $A_{obt.} - B_{obt.}$, with decreasing frequencies for the series: 9 ($A_{obt.} - B_{obt.} + 1$), 10 ($A_{obt.} - B_{obt.} + 2$), etc.; and for the series: 7 ($A_{obt.} - B_{obt} - 1$), 6 ($A_{obt.} + B_{obt.} - 2$), etc.

In general it can be shown that the divergence of the probable $A_{t.} - B_{t.}$ from $A_{obt.} - B_{obt.}$ is a variable fact, with a mode at zero, and a variability dependent on the variabilities of the divergences of $A_{t.}$ from $A_{obt.}$ and of $B_{t.}$ from $B_{obt.}$. The greater they are, the greater it is. It can be shown further that the form of the distribution of $A_{t.} - B_{t.}$, and so of the divergence of $A_{t.} - B_{t.}$ from $A_{obt.} - B_{obt.}$, is approximately of Form A, if the divergences $A_{t.} - A_{obt.}$ and $B_{t.} - B_{obt.}$ are approximately of Form A. It can be shown further that the variability of the probable divergence of $A_{t.} - B_{t.}$ from $A_{obt.} - B_{obt.}$ equals the square root of the sum of the squares of the probable divergences of $A_{t.}$ from $A_{obt.}$ and of $B_{t.}$ from $B_{obt.}$. That is,

$$\sigma_{(A_{t.}-B_{t.})-(A_{obt.}-B_{obt.})} = \sqrt{(\sigma_{(A_{t.}-A_{obt.})})^2 + (\sigma_{(B_{t.}-B_{obt.})})^2}$$

or

$$\sigma_{(t.-obt.\ Diff.\ A-B)} = \sqrt{(\sigma_{t.-obt.\ A})^2 + (\sigma_{t.-obt.\ B})^2}$$

The unreliability of a difference between A and B equals the square root of the sum of the square of the unreliability of A and the square of the unreliability of B.

The Unreliability of a Coefficient of Correlation.—The probable divergence of the true coefficient of correlation from that obtained from a limited random selection of the related pairs, is a variable fact with a mode at 0, and a variability which serves as the measure of the unreliability.

For r calculated from

$$\frac{\Sigma xy}{\sqrt{\Sigma x^2}\sqrt{\Sigma y^2}}, \text{ we may use } \sigma_{\text{t. } r-\text{obt. } r} = \frac{1-r^2}{\sqrt{n}}.$$

For r calculated from

$$\sqrt{[(\text{mid } x/y)v_1][(\text{mid } y/x)v_2]}, \text{ we may use } \sigma_{\text{t. } r-\text{obt. } r} = \frac{5}{4}\cdot\frac{1-r^2}{\sqrt{n}}.$$

For r calculated from

$$\cos \pi U, \text{ we may use } \sigma_{\text{t. } r-\text{obt. } r} = \frac{1.63}{\sqrt{n}}.$$

For r calculated from

$$2 \sin\left(\frac{\pi}{6}\rho\right), \text{ we may use } \sigma_{\text{t. } r-\text{obt. } r} = \frac{1.05(1-r^2)}{\sqrt{n}}.$$

Transmuting $\sigma_{\text{t.}-\text{obt.}}$'s into P.E.$_{\text{t.}-\text{obt.}}$'s.—So far the variability of a divergence of a true from an obtained measure has been expressed always as a mean square deviation. Since the distribution of the divergence will, if n is fairly large, approximate closely to Form *A*, the Med. Dev. or P.E. of the probable divergence of a true from an obtained measure may be taken as .6745[15] times the mean square deviation (S.D. or σ) of the same divergence. Similarly, the A.D. or M.V. of the probable divergence of a true from an obtained measure may be taken as .7979 times the mean square deviation of the same divergence.

We have then

$$\text{P.E.}_{\text{t. av.}-\text{obt. av.}} = .6745\frac{\sigma_{\text{dis.}}}{\sqrt{n}},$$

$$\text{P.E.}_{\text{t. med.}-\text{obt. med.}} = .6745\left(\frac{5}{4}\cdot\frac{\sigma_{\text{dis.}}}{\sqrt{n}}\right),$$

$$\text{P.E.}_{\text{t. }\sigma-\text{obt. }\sigma} = .6745\frac{\sigma_{\text{dis.}}}{\sqrt{2n}},$$

etc., etc.

Calculating Unreliabilities without Knowledge of $\sigma_{\text{dis.}}$.—In the case of distributions, with undistributed measures at either extreme, and in certain other cases, it is impossible to calculate $\sigma_{\text{dis.}}$. The unreliability of the obtained C.T. and variability in such a case may be estimated from the following:

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[15] More exactly .67449.

$$\sigma_{\text{t. av.--obt. av.}} = 1.4826\frac{Q_{\text{dis.}}}{\sqrt{n}},\quad \text{P.E.}_{\text{t. av.--obt. av.}}\,[16] = \frac{Q_{\text{dis.}}}{\sqrt{n}},$$

$$\sigma_{\text{t. med.--obt. med.}} = \frac{5}{4}\left(1.4826\frac{Q_{\text{dis.}}}{\sqrt{n}}\right),\quad \text{P.E.}_{\text{t. med.--obt. med.}} = \frac{5}{4}\frac{Q_{\text{dis.}}}{\sqrt{n}},$$

$$\sigma_{\text{t. }Q\text{--obt. }Q}\,[17] = \frac{1.65Q_{\text{dis.}}}{\sqrt{2n}},\quad \text{P.E.}_{\text{t. }Q\text{--obt. }Q} = \frac{1.11\,Q_{\text{dis.}}}{\sqrt{2n}},$$

$$\sigma_{(\text{t.--obt. Diff. }A-B)} = 1.4826\sqrt{(Q_{\text{t.--obt. }A})^2 + (Q_{\text{t.--obt. }B})^2},$$

$$\text{P.E.}_{(\text{t.--obt. Diff. }A-B)} = \sqrt{(Q_{\text{t.--obt. }A})^2 + (Q_{\text{t.--obt. }B})^2}.$$

PROBLEMS

What is the unreliability of each of the averages and of each of the σ's in the following cases?

47. $\text{Av.}_A = 10.$ $\sigma_{\text{dis.}} = 1.$ $N = 20.$

48. $\text{Av.}_B = 10.$ $\sigma_{\text{dis.}} = 1.5.$ $N = 30.$

49. $\text{Av.}_C = 12.$ $\sigma_{\text{dis.}} = 2.0.$ $N = 40.$

50. $\text{Av.}_D = 13.$ $\sigma_{\text{dis.}} = 3.0.$ $N = 40.$

51. $\text{Av.}_E = 14.$ $\sigma_{\text{dis.}} = 3.0.$ $N = 360.$

What is the unreliability of each of the following differences?

52. $\text{Av.}_C - \text{Av.}_A = 2.$ The data concerning Av._A and Av._C being as in 47 and 49.

53. $\text{Av.}_D - \text{Av.}_A = 3.$ The data concerning Av._A and Av._D being as in 47 and 50.

54. $\text{Av.}_E - \text{Av.}_A = 4.$ The data concerning Av._E and Av._A being as in 51 and 47.

55. $\text{Av.}_E - \text{Av.}_B = 4.$ The data concerning Av._E and Av._B being as in 51 and 48.

56. $\text{Av.}_E - \text{Av.}_C = 2.$ The data concerning Av._E and Av._C being as in 51 and 49.

What is the unreliability of r in each of the following cases, supposing r to have been calculated from

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[16] Q is, by tradition, not used as a name for a measure of unreliability $Q_{\text{t.--obt.}}$ would always be practically identical with $\text{P.E.}_{\text{t.--obt.}}$

[17] The Q in the subscript is, of course, the Q of the distribution in question.

$$\frac{\Sigma xy}{\sqrt{\Sigma x^2}\sqrt{\Sigma y^2}}?$$

57. $r = .46$. $N = 200$.
58. $r = .16$. $N = 200$.
59. $r = .16$. $N = 600$.

CHAPTER XIII

THE USE OF TABLES OF FREQUENCY OF THE PROBABILITY SURFACE

§ 45. *Tables of Values of the Normal Probability Integral*

TABLE 44 gives, for a surface of frequency of Form *A*,[1] the proportion of cases included between the average, 0, and various amounts of deviation therefrom, the latter being expressed as a multiple of the mean square deviation of the distribution.

Thus, the first line of entries of Table 44 reads: Between the average and $+ .01\sigma$, there are 40/10,000 or 0.4 per cent. of the cases; between the average and $+ .02\sigma$, there are 80/10,000 or .08 per cent. of the cases; between the average and $+ .03\sigma$, there are 120/10,000 or 1.2 per cent. of the cases, etc., etc. The facts are identical for $- .01\sigma$, $- .02\sigma$ and $- .03\sigma$.

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[1] The surface of frequency of Table 44 (which is that of a quantity due to the chance combinations of n causes, all equal and independent, when n is infinitely large) is, as has been stated elsewhere, the surface enclosed by the normal probability curve,

$$\left(Y = Pe^{\frac{-x^2}{2npq}} \quad \text{or} \quad y = e^{-x^2},\right.$$

or some specialized form, as

$$\left. y = \frac{1}{\mu\sqrt{2\pi}}\, e^{\frac{-x^2}{2\mu^2}} \right)$$

and the abscissa or base line on which x is scaled.

In this form of distribution the Average, Median and Mode coincide, for y is the same for any given $-x$ as for the same $+x$, and is greatest when $x = 0$. Constant relations hold between the different measures of variability, viz:

$$\begin{aligned} \sigma &= 1.25331 \text{ A.D.} \\ \sigma &= 1.48256 \text{ P.E.} \\ \text{A.D.} &= .7979\sigma \\ \text{A.D.} &= 1.1843 \text{ P.E.} \\ \text{P.E.} &= .6745\sigma \\ \text{P.E.} &= .8453 \text{ A.D.} \end{aligned}$$

Between (Av. $- \sigma$) and (Av. $+ \sigma$) are 68.26 per cent. of the cases.
Between (Av. $-$ A.D.) and (Av. $+$ A.D.) are 57.5 per cent. of the cases.
Between (Av. $-$ P.E.) and (Av. $+$ P.E.) are 50 per cent. of the cases.

TABLE 44[1]

TABLE OF VALUES OF THE NORMAL PROBABILITY INTEGRAL CORRESPONDING TO VALUES OF x/σ; *i. e.* THE FRACTION OF THE AREA OF THE SURFACE OF FREQUENCY OF FORM *A* BETWEEN THE LIMITS 0 AND $+x/\sigma$ OR 0 AND $-x/\sigma$

Total area of surface assumed to be 10,000. 100 = 1 *per cent.*

x = *deviation from mean.* σ = *standard deviation.*

x/σ	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	Δ
0.0	0000	0040	0080	0120	0160	0199	0239	0279	0319	0359	39.8
0.1	0398	0438	0478	0517	0557	0596	0636	0675	0714	0753	39.4
0.2	0793	0832	0871	0910	0948	0987	1026	1064	1103	1141	38.6
0.3	1179	1217	1255	1293	1331	1368	1406	1443	1480	1517	37.5
0.4	1554	1591	1628	1664	1700	1736	1772	1808	1844	1879	36.0
0.5	1915	1950	1985	2019	2054	2088	2123	2157	2190	2224	34.2
0.6	2257	2291	2324	2357	2389	2422	2454	2486	2517	2549	32.2
0.7	2580	2611	2642	2673	2704	2734	2764	2794	2823	2852	30.0
0.8	2881	2910	2939	2967	2995	3023	3051	3078	3106	3133	27.7
0.9	3159	3186	3212	3238	3264	3290	3315	3340	3365	3389	25.3
1.0	3413	3438	3461	3485	3508	3531	3554	3577	3599	3621	22.9
1.1	3643	3665	3686	3708	3729	3749	3770	3790	3810	3830	20.5
1.2	3849	3869	3888	3907	3925	3944	3962	3980	3997	4015	18.2
1.3	4032	4049	4066	4082	4099	4115	4131	4147	4162	4177	15.9
1.4	4192	4207	4222	4236	4251	4265	4279	4292	4306	4319	13.8
1.5	4332	4345	4357	4370	4383	4394	4406	4418	4429	4441	11.9
1.6	4452	4463	4474	4484	4495	4505	4515	4525	4535	4545	10.1
1.7	4554	4564	4573	4582	4591	4599	4608	4616	4625	4633	8.6
1.8	4641	4649	4656	4664	4671	4678	4686	4693	4699	4706	7.1
1.9	4713	4719	4726	4732	4738	4744	4750	4756	4761	4767	5.9
2.0	4772	4778	4783	4788	4793	4798	4803	4808	4812	4817	4.8
2.1	4821	4826	4830	4834	4838	4842	4846	4850	4854	4857	3.9
2.2	4861	4864	4868	4871	4875	4878	4881	4884	4887	4890	3.1
2.3	4893	4896	4898	4901	4904	4906	4909	4911	4913	4916	2.5
2.4	4918	4920	4922	4925	4927	4929	4931	4932	4934	4936	2.0
2.5	4938	4940	4941	4943	4945	4946	4948	4949	4951	4952	1.5
2.6	4953	4955	4956	4957	4959	4960	4961	4962	4963	4964	1.2
2.7	4965	4966	4967	4968	4969	4970	4971	4972	4973	4974	1.0
2.8	4974	4975	4976	4977	4977	4978	4979	4979	4980	4981	0.7
2.9	4981	4982	4982	4983	4984	4984	4985	4985	4986	4986	0.5
3.0	4986.5	4986.9	4987.4	4987.8	4988.2	4988.6	4988.9	4989.3	4989.7	4990.0	0.37
3.1	4990.3	4990.6	4991.0	4991.3	4991.6	4991.8	4992.1	4992.4	4992.6	4992.9	0.27
3.2	4993.129										0.17
3.3	4995.166										0.12
3.4	4996.631										0.10
3.5	4997.674										0.07
3.6	4998.409										0.05
3.7	4998.922										0.03
3.8	4999.277										0.02
3.9	4999.519										
4.0	4999.683										
4.5	4999.966										
5.0	4999.997133										

Any entry of the table, that is, gives the area bounded by (1) the base-line, (2) the boundary-line of the surface of frequency, (3) the vertical erected at the average and (4) the vertical erected at a given distance from the average. It gives the area as so many ten thousandths of the area of the entire surface. (1), (2) and (3) are the same for all entries. (4) is defined in the table by the "distance from the average," this distance being expressed as a multiple of σ. The table is for distances from the average up to 5.0σ, only three ten-millionths of the area being beyond that limit.

The table may be used for "distance *plus* from the average" or for "distance *minus* from the average" since the surface of Form *A* is symmetrical. The table has entries corresponding to the distances $.01\sigma$, $.02\sigma$, $.03\sigma$, etc., up to 3.20σ, and thereafter entries for 3.3σ, 3.4σ, 3.5σ, etc., up to 4.0σ, and for 4.5σ and 5.0σ.

The entries in the column under Δ give the approximate differences between neighboring entries in the body of the table, so as to allow convenient estimates of values corresponding to such distances from the average as $.036\sigma$, 057σ, 024σ and the like. Thus to find the proportion of the total area between the central tendency and $+ 1.464\sigma$ we take the table entry for 1.46 (which is 4279) and add $.4 \times 13.8$ (5.52), getting 4285.

Table 45 is the same as Table 44, except that (1) the distances from the central tendency are in multiples of the P.E. or *Q*, and that (2) the table is not so full, containing entries for only .1 P.E., .15 P.E., .20 P.E., .25 P.E., etc.

Table 46, in which the A.D. is the unit in which the distances are expressed, is still more abbreviated, containing entries for only .1 A.D., .2 A.D., .3 A.D., etc., and being for thousandths instead of ten thousandths of the total area.

[1] This table is arranged from the fuller table given by W. F. Sheppard in *Biometrika*, Vol. 2, pp. 182 ff.

TABLE 45

Table of Values of the Normal Probability Integral. The Fraction of the Area of the Surface of Frequency of Form *A* between the Limits, 0 and $+x/Q$ or 0 and $-x/Q$

Q = P. E.

Total area of surface assumed to be 10,000

x/Q	.00	.05	x/Q	.00	.05
0	000	135	3.0	4785	4802
.1	269	403	3.1	4817	4831
.2	536	670	3.2	4845	4858
.3	802	933	3.3	4870	4881
.4	1063	1193	3.4	4891	4900
.5	1321	1447	3.5	4909	4917
.6	1571	1695	3.6	4924	4931
.7	1816	1935	3.7	4937	4943
.8	2053	2168	3.8	4948	4953
.9	2291	2392	3.9	4957	4961
1.0	2500	2606	4.0	4965	4968
1.1	2709	2810	4.1	4971	4974
1.2	2908	3004	4.2	4977	4979
1.3	3097	3188	4.3	4981	4983
1.4	3275	3360	4.4	4985	4987
1.5	3441	3521	4.5	4988	4989
1.6	3597	3671	4.6	4990	4991
1.7	3742	3811	4.7	4992	4993
1.8	3896	3939	4.8	4994	4994.6
1.9	4000	4057	4.9	4995.2	4995.7
2.0	4113	4166	5.0	4996.2	4996.6
2.1	4217	4265	5.1	4997.1	4997.4
2.2	4311	4354	5.2	4997.7	4998.0
2.3	4396	4435	5.3	4998.2	4998.4
2.4	4472	4508	5.4	4998.6	4998.8
2.5	4541	4573	5.5	4999.0	4999.1
2.6	4602	4631	5.6	4999.2	4999.3
2.7	4657	4682	5.7	4999.4	4999.5
2.8	4705	4727	5.8	4999.55	4999.6
2.9	4748	4767	5.9	4999.65	4999.7

TABLE 46

Table of Values of the Normal Probability Integral Corresponding to Values of $x/(\text{A.D.})$

Total area of the surface of frequency taken as 1,000

x/A.D.	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0.	000	032	063	095	125	155	184	212	238	264
1.	288	310	331	350	368	384	399	413	425	435
2.	445	453	460	467	472	477	481	484	487	490
3.	492	493.4	494.6	495.8	496.7	497.4	498.0	498.4	498.7	499.1
4.	499.3	499.5	499.6	499.7	499.8	499.9				

To Reconstruct a Distribution from its Central Tendency and Variability, it Being of Form A.—Table 44 thus enables one to calculate the entire distribution of any trait which is "normally" distributed, if its central tendency and variability are known. For instance, if one finds for a given fact that the average = 24.0 and the mean square deviation = 4.0, one finds from the table that the ability 24 up to 25, or that between the average and $+ .25\sigma$, will be possessed by 9.87 per cent, of the group; the ability 24 up to 26, or that between 0 and $+ .5\sigma$ by 19.15 per cent., and consequently the ability 25 up to 26 by 19.15 − 9.87, or 9.28 per cent. By thus finding the percentages included between the average ability and different amounts of deviation from it, and so between any two given limits of deviation from it, one gets, as the table of frequencies in our illustrative case, Table 47.

TABLE 47

Relative Frequencies of a Variable Fact, Av. Being 24.0, σ being 4.0, and the Form of Distribution Being Form *A*

Quantity	Frequency	Quantity	Frequency
<11	0.06	24 up to 25	9.87
11 up to 12	0.07	25 up to 26	9.28
12 up to 13	0.17	26 up to 27	8.19
13 up to 14	0.32	27 up to 28	6.80
14 up to 15	0.60	28 up to 29	6.30
15 up to 16	1.05	29 up to 30	3.88
16 up to 17	1.73	30 up to 31	2.68
17 up to 18	2.68	31 up to 32	1.73
18 up to 19	3.88	32 up to 33	1.05
19 up to 20	6.30	33 up to 34	0.60
20 up to 21	6.80	34 up to 35	0.32
21 up to 22	8.19	35 up to 36	0.17
22 up to 23	9.28	36 up to 37	0.17
23 up to 24	9.87	>37	0.06

This use of the table gives a convenient means of measuring the degree to which the measures under investigation approximate in form to the probability distribution. If the table of actual frequencies of the measures is compared, entry for entry, with the frequencies given for corresponding deviations in the table for the probability surface, one can see at a glance the general closeness of correspondence. In making such comparisons, the actual frequencies may properly be grouped so as to represent only 18 or more grades, and any most likely central tendency may be chosen with which to make the central tendency of the probability surface coincide.

§ 46. *To Find the Percentage of Cases within Any Given Interval of the Scale*

The frequency of any degree of ability can obviously be calculated quickly if the central tendency and variability are given. For instance, if Av. = 10 and $\sigma = 2.4$, how many cases will be between 12.4 and 12.6? 12.4 is exactly 1σ from the Av. and 12.6 is 1.0833σ from the Av. The per cents. of cases included between Av. and 1σ and between Av. and 1.08σ are respectively 34.14 and 36.00. The number of cases between 1σ and 1.08σ is then 1.86 per cent. of the whole number in the series. To be exact and allow for the .0033, we add to the last figure one third of the difference in the table between the per cents. for 1.08 and 1.09, *viz.*, one third of a 22 or .0007. .3414 from .3607 then gives us .0193, or 1.93 per cent. The number of cases between 12.4 and 12.6 is, then, 1.93 per cent. of the whole number of cases. Practise with the following problems will familiarize one with this use of the tables:

60. Av. = 10. $\sigma = 3$. What per cent. of cases lie between 7 and 13.?

61. Av. = 22. $\sigma = 4.4$. What per cent. of cases lie between 18 and 20?

62. Av. = 15.5. $\sigma = 2.1$. What per cent. of cases lie above 22?

63. Av. = 15.5. $\sigma = 2.1$. What per cent. of cases lie below 13?

64. Av. = 14.86. $\sigma = 3.00$. What per cent. of cases lie between 12 and 13?

65. Av. = 14.86. $\sigma = 3.00$. What per cent. of cases lie between 14 and 16?

66. Av. = 29.74. P.E. = 3.18. What per cent. of cases lie between 24 and 25?

§ 47. *To find, from Any Starting-Point on the Scale, the Interval Required to Include a Given Percentage of the Cases*

By using the tables the other way about, one may find, Av. and variability being known, the degree of deviation from the average (or the distance from any stated point, *e. g.*, the upper limit, the lower limit, the point 1σ above the average, etc.) needed to include any stated percentage of the cases.

For instance, how far above the average must one go to get one fourth of the cases, the Av. being 8.0 and σ being 2.0? A distance of .67σ includes 2,486 and a distance of .68σ 2,518. A distance of .6745σ will obviously include 25 per cent. .6745 times 2 is 1.35. Hence the answer is 9.35. Again, what closest limits of ability will include 80 per cent. of the cases? From knowledge of the shape of the "normal" surface it is known that the cases are thickest the nearer they are to the average. So, we take, in the example, limits equidistant from the average. They are + 1.28σ and − 1.28σ, or more exactly, + 1.2817σ and − 1.2817σ. In the illustration these are 5.4366 and 10.5634. In reckoning inward from either extreme it is best to arbitrarily take 3σ as the limit plus or minus, though in the theoretical surface the limits are plus infinity and minus infinity.

The following are simple problems:

67. Av. = 10 and σ = 2. What limits will include the 30 per cent. just above the average?

68. The 20 per cent. below it?

69. The middle two thirds of the cases?

70. Av. = 17.24. σ = 4.0. What limits will include the middle three fourths of the cases?

71. The bottom 10 per cent.?

72. The second sixth of the cases from the top?

This use of the tables is that followed in transmuting a series of measures in terms of relative position into terms of amount. In so far as the distribution of the trait is that of the probability surface we can, calling the average 0, find the limits of deviation from it in terms of the variability as a unit which will include, say the lowest 1 per cent., the next 3 per cent., the 8 per cent. from the 23d to 31st per cent. from the top, etc. The process is so far identical with that in the examples just given. Then follows the calculation of an average amount to fit the cases included between each pair of limits. How this is done may be seen from a concrete case. Suppose that of 400 boys' themes 16, or 4 per cent., are indistinguishable for excellence, but are worse than 100 and better than 284. They are then per cents., 25, 26, 27 and 28. By Table *A* these per cents. will lie between + .6745σ and + .5531σ. By the table we find that the abilities between these limits have the following frequencies:

Ability	Frequency
.5531σ to .56σ	23
.56	34
.57	34
.58	34
.59	33
.60	33
.61	33
.62	33
.63	32
.64	33
.65	32
.66	32
.67σ to .6745σ	14

The average ability of the group is .61σ.

This is the method by which Tables 21 and 22 (pages 117 to 121) in Chapter VIII. were constructed.

§ 48. *To Find the Probability (P) that the Divergence of a True Measure from Its Corresponding Obtained Measure Will Be Within Any Given Limits*

The use of the tables here is the same as in § 46, the σ or P.E. in question being now the $\sigma_{\text{t.}-\text{obt. m.}}$ or P.E.$_{\text{t.}-\text{obt. m.}}$ instead of $\sigma_{\text{dis.}}$ or P.E.$_{\text{dis.}}$, and the central tendency in question being given as zero from the start.

For example, $\sigma_{\text{t. Av.}-\text{obt. Av.}}$ is 3.2. To find the chances that the true average will not vary from $A_{\text{obt.}}$ by more than 1.0, 2.0, 3.0, 4.0, 6.0 and 10.0. 1.0 is 31 per cent. of 3.2. By the table deviations within the limits + .31σ and − .31σ occur with a frequency of 12.17 + 12.17 or 24.34 per cent. There is, then, 1 chance out of 4 that $A_{\text{t.}}$ will not differ from $A_{\text{obt.}}$ by more than 1.0. 2.0 is 62.5 per cent. of 3.2. By the table deviations within the limits + .625σ and − .625σ occur in 46.8 per cent. of the cases. The chances are almost 1 to 1 that $A_{\text{t.}}$ will not differ from $A_{\text{obt.}}$ by more than 2.0. The chances of a difference of less than 10 will be found to be 9,982 out of 10,000 or over 550 to 1.

§ 49. *To Find, Starting from Zero, the Amount of Divergence of a True Measure from its Corresponding Obtained Measure Such that There Is a Given Probability that the Divergence in Question Will Be Less*

The use of the tables here is the same as in § 47, except that the σ or P.E. in question is, as in § 48, a measure of the variability of a divergence of a true from an obtained measure, and that the central tendency of this divergence is given as zero from the start.

For example, $\sigma_{\text{t. Av.—obt. Av.}}$ is 3.0. To find the amount of difference between $A_{\text{t.}}$ and $A_{\text{obt.}}$ differences greater than which will have only 1 chance in 100 of happening. In the table we find the distance from the average which must be passed over in both plus and minus directions to include 99 out of 100 cases, 49.5 plus and 49.5 minus. It is 2.575σ. Since σ equals 3.0 the answer to our problem is 7.725.

It will be noted that the tables serve equally well in the many cases where the desired fact is the probability of a given divergence *in one direction* or the amount of divergence *in one direction,* more divergence than which has a given degree of improbability.

The following problems will offer opportunity for acquiring self-confidence in the use of the tables in connection with all sorts of questions about unreliability:

73. $\text{Av.}_{\text{obt.}} = k$. $\sigma_{\text{t.—o. Av.}} = 1.6$. (*a*) What is the probability of a difference between $\text{Av.}_{\text{t.}}$ and $\text{Av.}_{\text{o.}}$ of 4.0 or more? (*b*) What are the chances that $\text{Av.}_{\text{t.}}$ will be 3.2 greater than $\text{Av.}_{\text{o.}}$? (*c*) Between what limits will the true average lie with a probability of 9999 to 1?

74. $\sigma_{\text{t.—o. var.}} = .4$. (*a*) What is the probability that the true variability is more than .8 less than the obtained? (*b*) That the true variability is not more than .6 above or below the obtained?

75. $\sigma_{\text{t.—o. diff.}} = .5$. The actually obtained difference is, $\text{Av.}_1 - \text{Av.}_2 = 1.2$. (*a*) What is the probability that the true difference is zero or less than zero? (*b*) That the true difference is: $\text{Av.}_1 - \text{Av.}_2 = 2.4$ or more? (*c*) That the true superiority of Av._1 over Av._2 is between 1.7 and .7? (*d*) What limits would you assign for the true difference to be sure that the chances would be 20 to 1 (*i. e.*, 20 in 21) against their being exceeded?

76. $r_{\text{o.}} = +.48$. $\sigma_{\text{t.—o. rel.}} = .04$. (*a*) Between what limits does the true relationship lie with practical certainty (it is customary to take 997 out of 1,000 as practical certainty)? (*b*) What is the chance that the true relationship is as low as .40?

77. $\text{Av.}_{\text{o.}} = 22.6$. $\sigma_{\text{t.—o. Av.}} = .5$. (*a*) What is the chance that the true average is as large as 24.0? (*b*) That it is as small as 22.0?

78. $\text{Av.}_{\text{o.}} = 28.2$. $\text{P.E.}_{\text{t.—o. Av.}} = .6$. (*a*) What is the chance that the true average is less than 26.0? (*b*) That it varies from $\text{Av.}_{\text{o.}}$ by less than 2.0?

79. If it were true that the chances were 82 to 18 that the true average would not vary from the obtained by more than 13.4, what would be the value of $\text{P.E.}_{\text{t.—o. Av.}}$?

80. $\text{Av.}_1 = 10.1$, $\text{Av.}_2 = 12.4$. $\text{P.E.}_{\text{t.—o. diff. of Av.}_1\text{ and Av.}_2} = 1.0$. (*a*) What are the chances that $\text{Av.}_2 - \text{Av.}_1 = 0$ or less? (*b*) 1.0 or less? (*c*) 2.5 or more? (*d*) Between 2.0 and 2.8? (*e*) Between 1.0 and 3.3?

81. $\text{P.E.}_{\text{dis. obt.}} = 1.6$, $\text{A.D.}_{\text{t.—o. var.}} = 0.1$. (*a*) What are the chances that $\text{P.E.}_{\text{dis.}}$ will be between 1.4 and 1.8? (*b*) That it will not exceed 1.9?

82. $r_{\text{o.}} = +.39$, $\text{P.E.}_{\text{t.—o. rel.}} = .008$. What is the chance of the true relationship being as high as + .40? As + .41? As + .42? As + .50?

83. Justify this statement from the tables.

> "Speaking roughly, the true measure is practically certain to lie between the following limits:
> Obtained measure $+ 3\sigma_{\text{t.—o. measure}}$ and obtained measure $-3\sigma_{\text{t.—o. measure}}$.
> Obtained measure + 4½ $\text{P.E.}_{\text{t.—o. measure}}$ and obtained measure −4½ $\text{P.E.}_{\text{t.—o. measure}}$."

84. $r_{1\text{o.}} - r_{2\text{o.}} = .04$, $\text{P.E.}_{\text{t.—o. diff. }r_1\text{ and }r_2} = .06$. (*a*) What is the chance that the true r_2 is really equal to or greater than the true r_1? (*b*) What is the chance that the true r_1 is greater than the true r_2?

CHAPTER XIV

SOURCES OF ERROR IN MEASUREMENTS

So far the supposition has been that the measures with which the calculations were made were accurate representatives of the fact measured, that *A* really did misspell the word which was scored as misspelled, that *B* did really take the .150 sec. to react which the chronoscope recorded, that the school enrollment and average attendance given for cities in the U. S. Commissioner's report gave the real facts, that the number of children recorded in certain genealogy books for certain families were the real numbers. The problem has been to make the best use of the data and introduce no error in manipulating them. But that a measure should thus perfectly represent a fact, the fact must be measured by a perfect instrument used by an infallible observer. In reality, any measure is a compound of a fact and the errors which the instrument and observer will surely make.

§ 50. *Variable Errors*

These errors may be *constant* or *variable*. A constant error is one tending more in one direction than the other. A watch that is too slow, a tendency of school superintendents to make the attendance record too high, are examples. Variable or chance errors are those tending in the long run to make the amount lower as often and as much as higher. The unevenness in action of a delicate balance due to dust, air currents, etc., the errors in addition made by the clerks in a superintendent's office, are examples.

Variable errors do not make any measure unfair, but only less exact and less reliable. If a body is weighed by an instrument which fluctuates so as to give 156.1, 156.2, 156.3, 156.3, 156.3, 156.3, 156.4, 156.4 and 156.4 in nine measurements, but is known not to weigh too light or heavy, 156.3 is a true measure, but the 156.3 only means between 156.25 and 156.35 and there is a slight chance of its being 156.2 or 156.4 (about 1 chance in 500).

If, on the contrary, a body is weighed by an instrument which fluctuates so little as to give 156.298, 156.299, 156.300, 156.300, 156.300, 156.301, 156.301 and 156.301, and which is known not to weigh too light or heavy, the 156.300 means between 156.2995 and 156.3005 and there is now certainty that the measure is not so low as 156.2 or so high as 156.4. Indeed, there is certainty that it is between 156.298 and 156.302.

There is no great advantage in decreasing the amount of the variable error by using more delicate instruments or more care in observing, unless the precision and reliability thereby obtained can be preserved in the further use of the measurements. The advantage that there is consists in the moral and intellectual training one gets and in the possibility that the measures may later be used for purposes other than one expects.

If we wish to get *A*'s average error in trying to equal a 100-mm. line, measurements may be made with the aid of a glass to $\frac{1}{10}$ mm., but the variation between *A*'s separate trials is so great that the larger error due to measuring each line so roughly as into $\frac{1}{2}$ mms. is insignificant. Indeed, measurements to a millimeter really do as well. If we wish to compare the reaction time of 1,000 boys with that of 1,000 girls, the median of 10 times being taken for each individual, measures in hundredths of seconds will do as well as measurements in thousandths.

Much time may be wasted in refining measurements in cases where no advantage accrues. And much ignorance is shown by the many students who disparage all measurements that are subject to a large variable error. They either do not know or forget that the reliability of a measure is due to the number of cases as well as to their variability, and that in the more complex and subtle mental traits it is always practicable to increase the number of measurements, but often impossible to make them less subject to variable errors. They also forget that the natural and real variability of the fact itself is often so large as to make the variability due to errors of instruments and observation practically negligible.

§ 51. *Constant Errors*

Constant errors, on the other hand, *are never negligible.*

The errors we make in interpreting handwriting would not, in a comparison of 1,000 boys with 1,000 girls in spelling ability, be worth spending a day on, even if thereby one could rectify them all, but if the teachers of the girls pronounced the words more clearly and phonetically than those of the boys, it would be necessary to discuss the proper discount or give up all hopes of precision. That a genealogist by mistake sometimes writes 4 or 7 matters practically *nil* to the student of vital statistics, but the genealogist's constant tendency to omit more children than he adds because of the difficulty of getting complete family records, is of the utmost importance.

Increasing the number of measures has here no beneficial influence. In certain cases increasing the number of observers may, namely, when the constant error of one observer is offset by the constant error *in the opposite direction* of another observer. If, that is, there is an error of prejudice or tendency constant for any one observer, but varying in direction by chance among a group of observers, what is a constant error for one becomes a variable error for a group, and is no longer a source of misleading, but only of lessened reliability. For instance, if any one person, even an expert judge, should rank 100 men in order for morality or efficiency or intellect, the results would probably have a constant error due to the undue weight he would put upon certain evidence; but if we took the median of the rankings given by ten or twelve expert judges, the error would in the main be only a chance error, for the prejudice of one would offset the prejudice of another.

The sources of constant errors in mental measurements are so numerous and so specialized for different kinds of facts that it is impossible to forearm the student against them here. Skill in avoiding them is due to capacity and watchfulness far more than to knowledge of any formal rules. It is, however, practically wise to test any result which may be affected by some constant error by using different methods of measurement, and to examine the means of selecting cases for measurement with the utmost care. The tendency to bias or to blunder is much more likely to make one select unfair cases than to make one measure them unfairly.

There is also a source of error which is perhaps in strictness an error in inference, but which from another point of view may be regarded as an error in measurement and so as relevant to the topics of this book. In measuring, say the spelling ability of a number of individuals whom we wish to compare, we assume that the achievement of each is a measure of the spelling ability of each. But *A* and *B* may have been seated where they did not hear the words pronounced so well as did *C* and *D*. *E* and *F* may have had headaches, while *G* and *H* were cheerful and bright. There exist errors due in the first example to outer physical conditions and in the second to inner or psychological conditions. To compare *A*, *B*, *C*, etc., in spelling ability, every extrinsic condition influencing that ability should be alike for all. Otherwise we are led into errors, which may be called errors of inferring an ability *in abstracto* from its manifestation under particular conditions, or of measuring a fact with a constant error of condition. It will be simpler to treat separately errors due to physical conditions and errors due to mental conditions.

Errors due to physical conditions can be prevented by making the conditions identical, or turned into relatively harmless variable errors by measuring each individual a number of times under conditions chosen at random. It would seem at first sight best to make conditions identical wherever practicable. This rule probably does hold for physical measurements, but there are certain disadvantages in this procedure in mental measurements. Too much artificiality and restraint in conditions often lead to an unusual and perturbed state of mind in the person measured, such that the thing one measures is likely to be a thing which would never occur in the ordinary course of the person's life. Measuring precisely a fact which you do not want is worse that measuring inexactly the fact you do want.

For instance, measurements of spelling under the unequal conditions of a schoolroom would, in spite of them, be better than measurements from 10-year-olds made to stand one at a time in the sound-proof room of a laboratory with head exactly 50 centimeters from a phonograph which pronounced the words for them to spell. The last method would give identity of physical conditions, but would measure insensibility to strange surroundings and treatment and ability to attend to and interpret the phonograph's noises perhaps more than it would spelling ability.

Errors due to mental conditions can not be prevented with surety by making the conditions identical, for it is not in the power of the observer to control the mental conditions of the person measured. The best that can be done is to avoid any probable cause of differences in them and to take the subjects' reports as to what their mental conditions are. But mental conditions vary greatly even despite the apparent absence of causes for difference; and the reports of mental condition from untrained self-observers must be vague, subject to constant errors and always from a personal standard of comparison incommensurate with that of any other individual. Though *A* says, "I am tired," and *B* says, "I am not," their feelings of fatigue may be equal. We do not take untrained individuals' opinions as facts elsewhere in science, and have no right to do so here. The more reliable procedure would be to eliminate the influence of the variability of inner conditions by random choice from among them rather than to pretend to eliminate the variation itself.

It is also a fair question whether the attempt to make all the mental conditions except the one to be measured alike in the persons to be compared, does not commonly result in so much unnaturalness of the sort against which protest was made a page back, as to do more harm than good. Attempted restriction of mental conditions surely disturbs anybody even more than restriction of physical conditions.

Success in eliminating disturbing conditions is not attainable as a result of knowledge of any fixed rules, but only through a happy ingenuity in devising experiments, arranging observations and selecting data. We can, however, be careful, after securing the best measurements that we can, to distinguish sharply between the actual measurement of the fact under certain conditions, on the one hand, and on the other the inferences that we may be tempted to make about the fact in general or apart from those particular conditions. It is not undesirable to make inferences, but it is highly undesirable to confuse them with measurements or to leave them without critical scrutiny.

Much more might well be said with regard to the sources of error prevalent in studies of human nature, but the proper bounds of an introduction, not to the logic or general method of the mental sciences, but only to their statistical problems, have already been passed.

§ 52. *Weighting Results*

Different sources of information concerning any one quantity may give it differing amounts, and these sources may be of unequal reliability. It is, then, desirable to allow more weight to the more trustworthy sources in deciding what amount is the most probable for the quantity. For instance, if an expert in physical anthropology measured *A*'s head and scored his cephalic index .81, while an ordinary person scored it .80, we should choose the .81 rather than the .80, and, if we allowed something for each judgment, would perhaps take 80.8 as the figure, counting the anthropologist's result four times.

No care in weighting sources will do so much service as the elimination of constant errors; and ideally no source with a constant error unallowed for should have any place in determining a result. Any source may deserve weight because of either numerical or qualitative strength. Its numerical strength is as the square root of the number of cases whose study it represents. Weighting for quality is bound in practise to be largely arbitrary, but this is not a great misfortune, for the result will rarely be altered appreciably by such differences in the system of weighting as reasonably competent students would make. For instance, *A*, *B* and *C* with the same general problem use different methods and get as a certain correlation coefficient .60, .50 and .48 respectively. Suppose that we weight these sources 1, 1 and 1; 4, 4 and 5; 3, 4 and 5; and finally 4, 3 and 5. We have then, as the probable true coefficient, .5267, .5231, .5167 or .5250. Bowley gives a rule that is satisfactory for most cases that occur in practise, namely, to give your attention to eliminating constant errors and not to manipulating weights.[1] If results are weighted it is always well to give them in their unweighted form as well and leave the opportunity open for any critic to weight them as he judges proper.

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[1] "In calculating averages give all your care to making the items free from bias and leave the weights to take care of themselves." "Elements of Statistics," p. 118.

APPENDICES

APPENDIX I

REFERENCES FOR FURTHER STUDY

IT is desirable that the student who has been introduced to statistical methods should proceed to study samples of their concrete application to problems in the mental sciences and, in case he has the necessary mathematical interest and training, that he should study the abstract properties of different forms of distribution, the derivation of statistical formulæ, the mathematical theory of correlation, and other topics in statistical theory. The following list of references to studies in psychology and education in which modern methods have been more or less fully applied to concrete problems is restricted to a few which are known to be suitable for such students as will use this book. There are doubtless others, of equal, or possibly greater, instructiveness. The bibliographies given at the end of each chapter of *An Introduction to the Theory of Statistics*, by G. Udny Yule (London, 1911) and on pp. 148 to 152 of *The Essentials of Mental Measurement* by W. Brown (Cambridge, England, 1911) make up an adequate list of references on the theory of measurements. I do not repeat these bibliographies, since these two books themselves should be in the hands of all advanced students of the theory of measurements.

1. On the Perception of Small Differences. By G. S. Fullerton and J. McK. Cattell. No. 2 of the *Philosophical Series of the Publications of the University of Pennsylvania*, May, 1892. The University of Pennsylvania Press, Philadelphia.
2. The Application of Statistical Methods to the Problems of Psycho-physics. By F. M. Urban. Philadelphia, 1908.
3. The Essentials of Mental Measurement. By William Brown, Cambridge (England), 1911. (Part I. of this book should be read in connection with references 1 and 2; Part II. should be read in connection with references 5 and 6.)
4. The Judgment of Difference: with Special Reference to the Doctrine of the Threshold in the Case of Lifted Weights.

By Warner Brown. *University of California Publications in Psychology,* vol. 1, No. 1, Sept., 1910.

5. Die Korrelation zwischen verschiedenen geistigen Leistungsfähigkeiten. By C. Spearman and F. Krueger. *Zeitschrift für Psychologie,* vol. 44, pp. 50–114 (1906).
6. Experimental Tests of General Intelligence. By Cyril Burt. *British Journal of Psychology,* vol. 3, pp. 94–177 (1909).
7. Natural Inheritance. By Francis Galton, London, 1889. (Chapters 8 and 9.)
8. Statistics of American Psychologists. By J. McKeen Cattell. *American Journal of Psychology,* vol. 14, pp. 310–328 (1903).
9. A Statistical Study of Literary Merit. By F. L. Wells. *Archives of Psychology,* No. 7 (1907).
10. Changes in the Age of College Graduation. By W. S. Thomas. *Popular Science Monthly,* June, 1903. (Reprinted in the Report of the U. S. Commissioner of Education for 1902, vol. 2, pp. 2199–2208.)
11. City School Expenditures. By G. D. Strayer. 1905. (This and the two following references are Nos. 5, 6, and 41 of the *Teachers College, Columbia University Contributions to Education.*)
12. Some Fiscal Aspects of Public Education in American Cities. By E. C. Elliott. 1905.
13. The Social Composition of the Teaching Population. By L. D. Coffman. 1911.

APPENDIX II

AIDS IN COMPUTATION

ATTENTION has been called in the text to Crelle's *Rechentafeln* (which gives the products up to 1000 × 1000, and, reversing its use, the quotients for division by numbers from 1 to 1000, to three figures); and to *Barlow's Tables*, for the squares, cubes, square roots, cube roots and reciprocals of numbers to 10,000. In addition, the following will be useful: Peters, J., *Neue Rechentafeln für Multiplikation und Division.* (Gives products up to 100 × 10,000.) The publishers of these three books are, in order:—G. Reimer, Berlin; Spon and Chamberlain, New York; G. Reimer, Berlin,

This appendix repeats, for convenience, some of the tables given in the text, and contains also a *Multiplication Table to 100 × 100*, a *Table of Squares and Square Roots for Numbers 1 to 1000*, and a separate *Multiplication Table with 1, 4, 9, 16, etc., as Multiplicands.* These briefer tables economize time and reduce eyestrain, and should be used instead of Crelle's and Barlow's, when one is working with numbers within their limits.

	TABLE 48	TABLE 49	TABLE 50
	Relative Frequencies (in Percentages) Over Each Tenth of σ, in a Surface of Frequency of Form *A*	**Relative Frequencies (in Percentages) Over Each Tenth of σ, in a Surface of Frequency of Form *C***	**Relative Frequencies (in Percentages) Over Each Tenth of σ, in a Surface of Frequency of Form *D***
−4.2 σ to −4.1 σ	.001		
−4.1 σ to −4.0 σ	.001		
−4.0 σ to −3.9 σ	.002		
−3.9 σ to −3.8 σ	.002		
−3.8 σ to −3.7 σ	.004		
−3.7 σ to −3.6 σ	.005		
−3.6 σ to −3.5 σ	.007		
−3.5 σ to −3.4 σ	.010		
−3.4 σ to −3.3 σ	.015		
−3.3 σ to −3.2 σ	.02		
−3.2 σ to −3.1 σ	.03		
−3.1 σ to −3.0 σ	.04		
−3.0 σ to −2.9 σ	.05		
−2.9 σ to −2.8 σ	.07		
−2.8 σ to −2.7 σ	.09		
−2.7 σ to −2.6 σ	.12		
−2.6 σ to −2.5 σ	15		
−2.5 σ to −2.4 σ	.20		
−2.4 σ to −2.3 σ	.25		
−2.3 σ to −2.2 σ	.32		
−2.2 σ to −2.1 σ	.40		
−2.1 σ to −2.0 σ	.49		
−2.0 σ to −1.9 σ	.60		
−1.9 σ to −1.8 σ	.72		
−1.8 σ to −1.7 σ	.86		
−1.7 σ to −1.6 σ	1.02		
−1.6 σ to −1.5 σ	1.20		
−1.5 σ to −1.4 σ	1.39		
−1.4 σ to −1.3 σ	1.60	.01	
−1.3 σ to −1.2 σ	1.83	.10	
−1.2 σ to −1.1 σ	2.06	.28	
−1.1 σ to −1.0 σ	2.30	.60	
−1.0 σ to − .9 σ	2.54	1.14	
− .9 σ to − .8 σ	2.78	1.76	
− .8 σ to − .7 σ	3.01	2.38	
− .7 σ to − .6 σ	3.23	2.94	.03
− .6 σ to − .5 σ	3.43	3.49	1.38
− .5 σ to − .4 σ	3.60	3.99	3.34
− .4 σ to − .3 σ	3.75	4.36	4.50
− .3 σ to − .2 σ	3.87	4.63	5.34
− .2 σ to − .1 σ	3.94	4.80	5.85
− .1 σ to 0 σ	3.98	4.89	6.05

	TABLE 48 (*continued*) Relative Frequencies (in Percentages) Over Each Tenth of σ, in a Surface of Frequency of Form *A*	TABLE 49 (*continued*) Relative Frequencies (in Percentages) Over Each Tenth of σ, in a Surface of Frequency of Form *C*	TABLE 50 (*continued*) Relative Frequencies (in Percentages) Over Each Tenth of σ, in a Surface of Frequency of Form *D*
0σ to $+.1\sigma$	3.98	4.89	6.05
$+.1\sigma$ to $+.2\sigma$	3.94	4.80	5.92
$+.2\sigma$ to $+.3\sigma$	3.87	4.65	5.67
$+.3\sigma$ to $+.4\sigma$	3.75	4.45	5.36
$+.4\sigma$ to $+.5\sigma$	3.60	4.26	5.02
$+.5\sigma$ to $+.6\sigma$	3.43	4.06	4.65
$+.6\sigma$ to $+.7\sigma$	3.23	3.80	4.30
$+.7\sigma$ to $+.8\sigma$	3.01	3.52	3.94
$+.8\sigma$ to $+.9\sigma$	2.78	3.25	3.62
$+.9\sigma$ to $+1.0\sigma$	2.54	2.99	3.31
$+1.0\sigma$ to $+1.1\sigma$	2.30	2.73	3.00
$+1.1\sigma$ to $+1.2\sigma$	2.06	2.48	2.69
$+1.2\sigma$ to $+1.3\sigma$	1.83	2.24	2.41
$+1.3\sigma$ to $+1.4\sigma$	1.60	2.02	2.19
$+1.4\sigma$ to $+1.5\sigma$	1.39	1.81	1.95
$+1.5\sigma$ to $+1.6\sigma$	1.20	1.62	1.73
$+1.6\sigma$ to $+1.7\sigma$	1.02	1.43	1.55
$+1.7\sigma$ to $+1.8\sigma$	.86	1.26	1.37
$+1.8\sigma$ to $+1.9\sigma$	.72	1.12	1.19
$+1.9\sigma$ to $+2.0\sigma$	.60	.98	1.05
$+2.0\sigma$ to $+2.1\sigma$	.49	.87	.93
$+2.1\sigma$ to $+2.2\sigma$	.40	.77	.80
$+2.2\sigma$ to $+2.3\sigma$	.32	.67	.69
$+2.3\sigma$ to $+2.4\sigma$	.25	58	.60
$+2.4\sigma$ to $+2.5\sigma$	.20	.50	.52
$+2.5\sigma$ to $+2.6\sigma$	.15	.44	.46
$+2.6\sigma$ to $+2.7\sigma$	.12	.39	.39
$+2.7\sigma$ to $+2.8\sigma$	.09	.34	.33
$+2.8\sigma$ to $+2.9\sigma$	.07	.29	.27
$+2.9\sigma$ to $+3.0\sigma$	.05	.25	.23
$+3.0\sigma$ to $+3.1\sigma$	.04	.21	.21
$+3.1\sigma$ to $+3.2\sigma$	.03	.18	.19
$+3.2\sigma$ to $+3.3\sigma$	.02	.16	.17
$+3.3\sigma$ to $+3.4\sigma$	.015	.14	.15
$+3.4\sigma$ to $+3.5\sigma$	.010	.12	.14
$+3.5\sigma$ to $+3.6\sigma$	.007	.10	.12
$+3.6\sigma$ to $+3.7\sigma$	.005	.08	.10
$+3.7\sigma$ to $+3.8\sigma$	.004	.07	.08
$+3.8\sigma$ to $+3.9\sigma$	.002	.05	.07
$+3.9\sigma$ to $+4.0\sigma$	.002	.03	.05
$+4.0\sigma$ to $+4.1\sigma$	.001	.015	.03
$+4.1\sigma$ to $+4.2\sigma$	.001	.005	.015
$+4.2\sigma$ to $+4.3\sigma$			.005

TABLE 51

TABLE OF VALUES OF THE NORMAL PROBABILITY INTEGRAL CORRESPONDING TO VALUES OF x/σ; *i. e.* THE FRACTION OF THE AREA OF THE SURFACE OF FREQUENCY OF FORM *A* BETWEEN THE LIMITS 0 AND $+x/\sigma$ OR 0 AND $-x/\sigma$

Total area of surface assumed to be 10,000. 100 = 1 *per cent.*

x = *deviation from mean.* σ = *standard deviation.*

x/σ	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	Δ
0.0	0000	0040	0080	0120	0160	0199	0239	0279	0319	0359	39.8
0.1	0398	0438	0478	0517	0557	0596	0636	0675	0714	0753	39.4
0.2	0793	0832	0871	0910	0948	0987	1026	1064	1103	1141	38.6
0.3	1179	1217	1255	1293	1331	1368	1406	1443	1480	1517	37.5
0.4	1554	1591	1628	1664	1700	1736	1772	1808	1844	1879	36.0
0.5	1915	1950	1985	2019	2054	2088	2123	2157	2190	2224	34.2
0.6	2257	2291	2324	2357	2389	2422	2454	2486	2517	2549	32.2
0.7	2580	2611	2642	2673	2704	2734	2764	2794	2823	2852	30.0
0.8	2881	2910	2939	2967	2995	3023	3051	3078	3106	3133	27.7
0.9	3159	3186	3212	3238	3264	3290	3315	3340	3365	3389	25.3
1.0	3413	3438	3461	3485	3508	3531	3554	3577	3599	3621	22.9
1.1	3643	3665	3686	3708	3729	3749	3770	3790	3810	3830	20.5
1.2	3849	3869	3888	3907	3925	3944	3962	3980	3997	4015	18.2
1.3	4032	4049	4066	4082	4099	4115	4131	4147	4162	4177	15.9
1.4	4192	4207	4222	4236	4251	4265	4279	4292	4306	4319	13.8
1.5	4332	4345	4357	4370	4383	4394	4406	4418	4429	4441	11.9
1.6	4452	4463	4474	4484	4495	4505	4515	4525	4535	4545	10.1
1.7	4554	4564	4573	4582	4591	4599	4608	4616	4625	4633	8.6
1.8	4641	4649	4656	4664	4671	4678	4686	4693	4699	4706	7.1
1.9	4713	4719	4726	4732	4738	4744	4750	4756	4761	4767	5.9
2.0	4772	4778	4783	4788	4793	4798	4803	4808	4812	4817	4.8
2.1	4821	4826	4830	4834	4838	4842	4846	4850	4854	4857	3.9
2.2	4861	4864	4868	4871	4875	4878	4881	4884	4887	4890	3.1
2.3	4893	4896	4898	4901	4904	4906	4909	4911	4913	4916	2.5
2.4	4918	4920	4922	4925	4927	4929	4931	4932	4934	4936	2.0
2.5	4938	4940	4941	4943	4945	4946	4948	4949	4951	4952	1.5
2.6	4953	4955	4956	4957	4959	4960	4961	4962	4963	4964	1.2
2.7	4965	4966	4967	4968	4969	4970	4971	4972	4973	4974	1.0
2.8	4974	4975	4976	4977	4977	4978	4979	4979	4980	4981	0.7
2.9	4981	4982	4982	4983	4984	4984	4985	4985	4986	4986	0.5
3.0	4986.5	4986.9	4987.4	4987.8	4988.2	4988.6	4988.9	4989.3	4989.7	4990.0	0.37
3.1	4990.3	4990.6	4991.0	4991.3	4991.6	4991.8	4992.1	4992.4	4992.6	4992.9	0.27
3.2	4993.129										0.17
3.3	4995.166										0.12
3.4	4996.631										0.10
3.5	4997.674										0.07
3.6	4998.409										0.05
3.7	4998.922										0.03
3.8	4999.277										0.02
3.9	4999.519										
4.0	4999.683										
4.5	4999.966										
5.0	4999.997133										

	51	52	53	54	55	56	57	58	59	60	
1	51	52	53	54	55	56	57	58	59	60	1
2	102	104	106	108	110	112	114	116	118	120	2
3	153	156	159	162	165	168	171	174	177	180	3
4	204	208	212	216	220	224	228	232	236	240	4
5	255	260	265	270	275	280	285	290	295	300	5
6	306	312	318	324	330	336	342	348	354	360	6
7	357	364	371	378	385	392	399	406	413	420	7
8	408	416	424	432	440	448	456	464	472	480	8
9	459	468	477	486	495	504	513	522	531	540	9
10	510	520	530	540	550	560	570	580	590	600	10
11	561	572	583	594	605	616	627	638	649	660	11
12	612	624	636	648	660	672	684	696	708	720	12
13	663	676	689	702	715	728	741	754	767	780	13
14	714	728	742	756	770	784	798	812	826	840	14
15	765	780	795	810	825	840	855	870	885	900	15
16	816	832	848	864	880	896	912	928	944	960	16
17	867	884	901	918	935	952	969	986	1003	1020	17
18	918	936	954	972	990	1008	1026	1044	1062	1080	18
19	969	988	1007	1026	1045	1064	1083	1102	1121	1140	19
20	1020	1040	1060	1080	1100	1120	1140	1160	1180	1200	20
21	1071	1092	1113	1134	1155	1176	1197	1218	1239	1260	21
22	1122	1144	1166	1188	1210	1232	1254	1276	1298	1320	22
23	1173	1196	1219	1242	1265	1288	1311	1334	1357	1380	23
24	1224	1248	1272	1296	1320	1344	1368	1392	1416	1440	24
25	1275	1300	1325	1350	1375	1400	1425	1450	1475	1500	25
26	1326	1352	1378	1404	1430	1456	1482	1508	1534	1560	26
27	1377	1404	1431	1458	1485	1512	1539	1566	1593	1620	27
28	1428	1456	1484	1512	1540	1568	1596	1624	1652	1680	28
29	1479	1508	1537	1566	1595	1624	1653	1682	1711	1740	29
30	1530	1560	1590	1620	1650	1680	1710	1740	1770	1800	30
31	1581	1612	1643	1674	1705	1736	1767	1798	1829	1860	31
32	1632	1664	1696	1728	1760	1792	1824	1856	1888	1920	32
33	1683	1716	1749	1782	1815	1848	1881	1914	1947	1980	33
34	1734	1768	1802	1836	1870	1904	1938	1972	2006	2040	34
35	1785	1820	1855	1890	1925	1960	1995	2030	2065	2100	35
36	1836	1872	1908	1944	1980	2016	2052	2088	2124	2160	36
37	1887	1924	1961	1998	2035	2072	2109	2146	2183	2220	37
38	1938	1976	2014	2052	2090	2128	2166	2204	2242	2280	38
39	1989	2028	2067	2106	2145	2184	2223	2262	2301	2340	39
40	2040	2080	2120	2160	2200	2240	2280	2320	2360	2400	40
41	2091	2132	2173	2214	2255	2296	2337	2378	2419	2460	41
42	2142	2184	2226	2268	2310	2352	2394	2436	2478	2520	42
43	2193	2236	2279	2322	2365	2408	2451	2494	2537	2580	43
44	2244	2288	2332	2376	2420	2464	2508	2552	2596	2640	44
45	2295	2340	2385	2430	2475	2520	2565	2610	2655	2700	45
46	2346	2392	2438	2484	2530	2576	2622	2668	2714	2760	46
47	2397	2444	2491	2538	2585	2632	2679	2726	2773	2820	47
48	2448	2496	2544	2592	2640	2688	2736	2784	2832	2880	48
49	2499	2548	2597	2646	2695	2744	2793	2842	2891	2940	49
50	2550	2600	2650	2700	2750	2800	2850	2900	2950	3000	50
	51	52	53	54	55	56	57	58	59	60	

	51	52	53	54	55	56	57	58	59	60	
51	2601	2652	2703	2754	2805	2856	2907	2958	3009	3060	51
52	2652	2704	2756	2808	2860	2912	2964	3016	3068	3120	52
53	2703	2756	2809	2862	2915	2968	3021	3074	3127	3180	53
54	2754	2808	2862	2916	2970	3024	3078	3132	3186	3240	54
55	2805	2860	2915	2970	3025	3080	3135	3190	3245	3300	55
56	2856	2912	2968	3024	3080	3136	3192	3248	3304	3360	56
57	2907	2964	3021	3078	3135	3192	3249	3306	3363	3420	57
58	2958	3016	3074	3132	3190	3248	3306	3364	3422	3480	58
59	3009	3068	3127	3186	3245	3304	3363	3422	3481	3540	59
60	3060	3120	3180	3240	3300	3360	3420	3480	3540	3600	60
61	3111	3172	3233	3294	3355	3416	3477	3538	3599	3660	61
62	3162	3224	3286	3348	3410	3472	3534	3596	3658	3720	62
63	3213	3276	3339	3402	3465	3528	3591	3654	3717	3780	63
64	3264	3328	3392	3456	3520	3584	3648	3712	3776	3840	64
65	3315	3380	3445	3510	3575	3640	3705	3770	3835	3900	65
66	3366	3432	3498	3564	3630	3696	3762	3828	3894	3960	66
67	3417	3484	3551	3618	3685	3752	3819	3886	3953	4020	67
68	3468	3536	3604	3672	3740	3808	3876	3944	4012	4080	68
69	3519	3588	3657	3726	3795	3864	3933	4002	4071	4140	69
70	3570	3640	3710	3780	3850	3920	3990	4060	4130	4200	70
71	3621	3692	3763	3834	3905	3976	4047	4118	4189	4260	71
72	3672	3744	3816	3888	3960	4032	4104	4176	4248	4320	72
73	3723	3796	3869	3942	4015	4088	4161	4234	4307	4380	73
74	3774	3848	3922	3996	4070	4144	4218	4292	4366	4440	74
75	3825	3900	3975	4050	4125	4200	4275	4350	4425	4500	75
76	3876	3952	4028	4104	4180	4256	4332	4408	4484	4560	76
77	3927	4004	4081	4158	4235	4312	4389	4466	4543	4620	77
78	3978	4056	4134	4212	4290	4368	4446	4524	4602	4680	78
79	4029	4108	4187	4266	4345	4424	4503	4582	4661	4740	79
80	4080	4160	4240	4320	4400	4480	4560	4640	4720	4800	80
81	4131	4212	4293	4374	4455	4536	4617	4698	4779	4860	81
82	4182	4264	4346	4428	4510	4592	4674	4756	4838	4920	82
83	4233	4316	4399	4482	4565	4648	4731	4814	4897	4980	83
84	4284	4368	4452	4536	4620	4704	4788	4872	4956	5040	84
85	4335	4420	4505	4590	4675	4760	4845	4930	5015	5100	85
86	4386	4472	4558	4644	4730	4816	4902	4988	5074	5160	86
87	4437	4524	4611	4698	4785	4872	4959	5046	5133	5220	87
88	4488	4576	4664	4752	4840	4928	5016	5104	5192	5280	88
89	4539	4628	4717	4806	4895	4984	5073	5162	5251	5340	89
90	4590	4680	4770	4860	4950	5040	5130	5220	5310	5400	90
91	4641	4732	4823	4914	5005	5096	5187	5278	5369	5460	91
92	4692	4784	4876	4968	5060	5152	5244	5336	5428	5520	92
93	4743	4836	4929	5022	5115	5208	5301	5394	5487	5580	93
94	4794	4888	4982	5076	5170	5264	5358	5452	5546	5640	94
95	4845	4940	5035	5130	5225	5320	5415	5510	5605	5700	95
96	4896	4992	5088	5184	5280	5376	5472	5568	5664	5760	96
97	4947	5044	5141	5238	5335	5432	5529	5626	5723	5820	97
98	4998	5096	5194	5292	5390	5488	5586	5684	5782	5880	98
99	5049	5148	5247	5346	5445	5544	5643	5742	5841	5940	99
100	5100	5200	5300	5400	5500	5600	5700	5800	5900	6000	100
	51	52	53	54	55	56	57	58	59	60	

	61	62	63	64	65	66	67	68	69	70	
1	61	62	63	64	65	66	67	68	69	70	1
2	122	124	126	128	130	132	134	136	138	140	2
3	183	186	189	192	195	198	201	204	207	210	3
4	244	248	252	256	260	264	268	272	276	280	4
5	305	310	315	320	325	330	335	340	345	350	5
6	366	372	378	384	390	396	402	408	414	420	6
7	427	434	441	448	455	462	469	476	483	490	7
8	488	496	504	512	520	528	536	544	552	560	8
9	549	558	567	576	585	594	603	612	621	630	9
10	610	620	630	640	650	660	670	680	690	700	10
11	671	682	693	704	715	726	737	748	759	770	11
12	732	744	756	768	780	792	804	816	828	840	12
13	793	806	819	832	845	858	871	884	897	910	13
14	854	868	882	896	910	924	938	952	966	980	14
15	915	930	945	960	975	990	1005	1020	1035	1050	15
16	976	992	1008	1024	1040	1056	1072	1088	1104	1120	16
17	1037	1054	1071	1088	1105	1122	1139	1156	1173	1190	17
18	1098	1116	1134	1152	1170	1188	1206	1224	1242	1260	18
19	1159	1178	1197	1216	1235	1254	1273	1292	1311	1330	19
20	1220	1240	1260	1280	1300	1320	1340	1360	1380	1400	20
21	1281	1302	1323	1344	1365	1386	1407	1428	1449	1470	21
22	1342	1364	1386	1408	1430	1452	1474	1496	1518	1540	22
23	1403	1426	1449	1472	1495	1518	1541	1564	1587	1610	23
24	1464	1488	1512	1536	1560	1584	1608	1632	1656	1680	24
25	1525	1550	1575	1600	1625	1650	1675	1700	1725	1750	25
26	1586	1612	1638	1664	1690	1716	1742	1768	1794	1820	26
27	1647	1674	1701	1728	1755	1782	1809	1836	1863	1890	27
28	1708	1736	1764	1792	1820	1848	1876	1904	1932	1960	28
29	1769	1798	1827	1856	1885	1914	1943	1972	2001	2030	29
30	1830	1860	1890	1920	1950	1980	2010	2040	2070	2100	30
31	1891	1922	1953	1984	2015	2046	2077	2108	2139	2170	31
32	1952	1984	2016	2048	2080	2112	2144	2176	2208	2240	32
33	2013	2046	2079	2112	2145	2178	2211	2244	2277	2310	33
34	2074	2108	2142	2176	2210	2244	2278	2312	2346	2380	34
35	2135	2170	2205	2240	2275	2310	2345	2380	2415	2450	35
36	2196	2232	2268	2304	2340	2376	2412	2448	2484	2520	36
37	2257	2294	2331	2368	2405	2442	2479	2516	2553	2590	37
38	2318	2356	2394	2432	2470	2508	2546	2584	2622	2660	38
39	2379	2418	2457	2496	2535	2574	2613	2652	2691	2730	39
40	2440	2480	2520	2560	2600	2640	2680	2720	2760	2800	40
41	2501	2542	2583	2624	2665	2706	2747	2788	2829	2870	41
42	2562	2604	2646	2688	2730	2772	2814	2856	2898	2940	42
43	2623	2666	2709	2752	2795	2838	2881	2924	2967	3010	43
44	2684	2728	2772	2816	2860	2904	2948	2992	3036	3080	44
45	2745	2790	2835	2880	2925	2970	3015	3060	3105	3150	45
46	2806	2852	2898	2944	2990	3036	3082	3128	3174	3220	46
47	2867	2914	2961	3008	3055	3102	3149	3196	3243	3290	47
48	2928	2976	3024	3072	3120	3168	3216	3264	3312	3360	48
49	2989	3038	3087	3136	3185	3234	3283	3332	3381	3430	49
50	3050	3100	3150	3200	3250	3300	3350	3400	3450	3500	50
	61	62	63	64	65	66	67	68	69	70	

	61	62	63	64	65	66	67	68	69	70	
51	3111	3162	3213	3264	3315	3366	3417	3468	3519	3570	51
52	3172	3224	3276	3328	3380	3432	3484	3536	3588	3640	52
53	3233	3286	3339	3392	3445	3498	3551	3604	3657	3710	53
54	3294	3348	3402	3456	3510	3564	3618	3672	3726	3780	54
55	3355	3410	3465	3520	3575	3630	3685	3740	3795	3850	55
56	3416	3472	3528	3584	3640	3696	3752	3808	3864	3920	56
57	3477	3534	3591	3648	3705	3762	3819	3876	3933	3990	57
58	3538	3596	3654	3712	3770	3828	3886	3944	4002	4060	58
59	3599	3658	3717	3776	3835	3894	3953	4012	4071	4130	59
60	3660	3720	3780	3840	3900	3960	4020	4080	4140	4200	60
61	3721	3782	3843	3904	3965	4026	4087	4148	4209	4270	61
62	3782	3844	3906	3968	4030	4092	4154	4216	4278	4340	62
63	3843	3906	3969	4032	4095	4158	4221	4284	4347	4410	63
64	3904	3968	4032	4096	4160	4224	4288	4352	4416	4480	64
65	3965	4030	4095	4160	4225	4290	4355	4420	4485	4550	65
66	4026	4092	4158	4224	4290	4356	4422	4488	4554	4620	66
67	4087	4154	4221	4288	4355	4422	4489	4556	4623	4690	67
68	4148	4216	4284	4352	4420	4488	4556	4624	4692	4760	68
69	4209	4278	4347	4416	4485	4554	4623	4692	4761	4830	69
70	4270	4340	4410	4480	4550	4620	4690	4760	4830	4900	70
71	4331	4402	4473	4544	4615	4686	4757	4828	4899	4970	71
72	4392	4464	4536	4608	4680	4752	4824	4896	4968	5040	72
73	4453	4526	4599	4672	4745	4818	4891	4964	5037	5110	73
74	4514	4588	4662	4736	4810	4884	4958	5032	5106	5180	74
75	4575	4650	4725	4800	4875	4950	5025	5100	5175	5250	75
76	4636	4712	4788	4864	4940	5016	5092	5168	5244	5320	76
77	4697	4774	4851	4928	5005	5082	5159	5236	5313	5390	77
78	4758	4836	4914	4992	5070	5148	5226	5304	5382	5460	78
79	4819	4898	4977	5056	5135	5214	5293	5372	5451	5530	79
80	4880	4960	5040	5120	5200	5280	5360	5440	5520	5600	80
81	4941	5022	5103	5184	5265	5346	5427	5508	5589	5670	81
82	5002	5084	5166	5248	5330	5412	5494	5576	5658	5740	82
83	5063	5146	5229	5312	5395	5478	5561	5644	5727	5810	83
84	5124	5208	5292	5376	5460	5544	5628	5712	5796	5880	84
85	5185	5270	5355	5440	5525	5610	5695	5780	5865	5950	85
86	5246	5332	5418	5504	5590	5676	5762	5848	5934	6020	86
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88	5368	5456	5544	5632	5720	5808	5896	5984	6072	6160	88
89	5429	5518	5607	5696	5785	5874	5963	6052	6141	6230	89
90	5490	5580	5670	5760	5850	5940	6030	6120	6210	6300	90
91	5551	5642	5733	5824	5915	6006	6097	6188	6279	6370	91
92	5612	5704	5796	5888	5980	6072	6164	6256	6348	6440	92
93	5673	5766	5859	5952	6045	6138	6231	6324	6417	6510	93
94	5734	5828	5922	6016	6110	6204	6298	6392	6486	6580	94
95	5795	5890	5985	6080	6175	6270	6365	6460	6555	6650	95
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7	497	504	511	518	525	532	539	546	553	560	7
8	568	576	584	592	600	608	616	624	632	640	8
9	639	648	657	666	675	684	693	702	711	720	9
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27	1917	1944	1971	1998	2025	2052	2079	2106	2133	2160	27
28	1988	2016	2044	2072	2100	2128	2156	2184	2212	2240	28
29	2059	2088	2117	2146	2175	2204	2233	2262	2291	2320	29
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31	2201	2232	2263	2294	2325	2356	2387	2418	2449	2480	31
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42	2982	3024	3066	3108	3150	3192	3234	3276	3318	3360	42
43	3053	3096	3139	3182	3225	3268	3311	3354	3397	3440	43
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52	3692	3744	3796	3848	3900	3952	4004	4056	4108	4160	**52**
53	3763	3816	3869	3922	3975	4028	4081	4134	4187	4240	**53**
54	3834	3888	3942	3996	4050	4104	4158	4212	4266	4320	**54**
55	3905	3960	4015	4070	4125	4180	4235	4290	4345	4400	**55**
56	3976	4032	4088	4144	4200	4256	4312	4368	4424	4480	**56**
57	4047	4104	4161	4218	4275	4332	4389	4446	4503	4560	**57**
58	4118	4176	4234	4292	4350	4408	4466	4524	4582	4640	**58**
59	4189	4248	4307	4366	4425	4484	4543	4602	4661	4720	**59**
60	4260	4320	4380	4440	4500	4560	4620	4680	4740	4800	**60**
61	4331	4392	4453	4514	4575	4636	4697	4758	4819	4880	**61**
62	4402	4464	4526	4588	4650	4712	4774	4836	4898	4960	**62**
63	4473	4536	4599	4662	4725	4788	4851	4914	4977	5040	**63**
64	4544	4608	4672	4736	4800	4864	4928	4992	5056	5120	**64**
65	4615	4680	4745	4810	4875	4940	5005	5070	5135	5200	**65**
66	4686	4752	4818	4884	4950	5016	5082	5148	5214	5280	**66**
67	4757	4824	4891	4958	5025	5092	5159	5226	5293	5360	**67**
68	4828	4896	4964	5032	5100	5168	5236	5304	5372	5440	**68**
69	4899	4968	5037	5106	5175	5244	5313	5382	5451	5520	**69**
70	4970	5040	5110	5180	5250	5320	5390	5460	5530	5600	**70**
71	5041	5112	5183	5254	5325	5396	5467	5538	5609	5680	**71**
72	5112	5184	5256	5328	5400	5472	5544	5616	5688	5760	**72**
73	5183	5256	5329	5402	5475	5548	5621	5694	5767	5840	**73**
74	5254	5328	5402	5476	5550	5624	5698	5772	5846	5920	**74**
75	5325	5400	5475	5550	5625	5700	5775	5850	5925	6000	**75**
76	5396	5472	5548	5624	5700	5776	5852	5928	6004	6080	**76**
77	5467	5544	5621	5698	5775	5852	5929	6006	6083	6160	**77**
78	5538	5616	5694	5772	5850	5928	6006	6084	6162	6240	**78**
79	5609	5688	5767	5846	5925	6004	6083	6162	6241	6320	**79**
80	5680	5760	5840	5920	6000	6080	6160	6240	6320	6400	**80**
81	5751	5832	5913	5994	6075	6156	6237	6318	6399	6480	**81**
82	5822	5904	5986	6068	6150	6232	6314	6396	6478	6560	**82**
83	5893	5976	6059	6142	6225	6308	6391	6474	6557	6640	**83**
84	5964	6048	6132	6216	6300	6384	6468	6552	6636	6720	**84**
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87	6177	6264	6351	6438	6525	6612	6699	6786	6873	6960	**87**
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89	6319	6408	6497	6586	6675	6764	6853	6942	7031	7120	**89**
90	6390	6480	6570	6660	6750	6840	6930	7020	7110	7200	**90**
91	6461	6552	6643	6734	6825	6916	7007	7098	7189	7280	**91**
92	6532	6624	6716	6808	6900	6992	7084	7176	7268	7360	**92**
93	6603	6696	6789	6882	6975	7068	7161	7254	7347	7440	**93**
94	6674	6768	6862	6956	7050	7144	7238	7332	7426	7520	**94**
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97	6887	6984	7081	7178	7275	7372	7469	7566	7663	7760	**97**
98	6958	7056	7154	7252	7350	7448	7546	7644	7742	7840	**98**
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3	243	246	249	252	255	258	261	264	267	270	3
4	324	328	332	336	340	344	348	352	356	360	4
5	405	410	415	420	425	430	435	440	445	450	5
6	486	492	498	504	510	516	522	528	534	540	6
7	567	574	581	588	595	602	609	616	623	630	7
8	648	656	664	672	680	688	696	704	712	720	8
9	729	738	747	756	765	774	783	792	801	810	9
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27	2187	2214	2241	2268	2295	2322	2349	2376	2403	2430	27
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36	2916	2952	2988	3024	3060	3096	3132	3168	3204	3240	36
37	2997	3034	3071	3108	3145	3182	3219	3256	3293	3330	37
38	3078	3116	3154	3192	3230	3268	3306	3344	3382	3420	38
39	3159	3198	3237	3276	3315	3354	3393	3432	3471	3510	39
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43	3483	3526	3569	3612	3655	3698	3741	3784	3827	3870	43
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57	4617	4674	4731	4788	4845	4902	4959	5016	5073	5130	57
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59	4779	4838	4897	4956	5015	5074	5133	5192	5251	5310	59
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61	4941	5002	5063	5124	5185	5246	5307	5368	5429	5490	61
62	5022	5084	5146	5208	5270	5332	5394	5456	5518	5580	62
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66	5346	5412	5478	5544	5610	5676	5742	5808	5874	5940	66
67	5427	5494	5561	5628	5695	5762	5829	5896	5963	6030	67
68	5508	5576	5644	5712	5780	5848	5916	5984	6052	6120	68
69	5589	5658	5727	5796	5865	5934	6003	6072	6141	6210	89
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71	5751	5822	5893	5964	6035	6106	6177	6248	6319	6390	71
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73	5913	5986	6059	6132	6205	6278	6351	6424	6497	6570	73
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83	6723	6806	6889	6972	7055	7138	7221	7304	7387	7470	83
84	6804	6888	6972	7056	7140	7224	7308	7392	7476	7560	84
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86	6966	7052	7138	7224	7310	7396	7482	7568	7654	7740	86
87	7047	7134	7221	7308	7395	7482	7569	7656	7743	7830	87
88	7128	7216	7304	7392	7480	7568	7656	7744	7832	7920	88
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94	7614	7708	7802	7896	7990	8084	8178	8272	8366	8460	94
95	7695	7790	7885	7980	8075	8170	8265	8360	8455	8550	95
96	7776	7872	7968	8064	8160	8256	8352	8448	8544	8640	96
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7	637	644	651	658	665	672	679	686	693	700	7
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9	819	828	837	846	855	864	873	882	891	900	9
10	910	920	930	940	950	960	970	980	990	1000	10
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16	1456	1472	1488	1504	1520	1536	1552	1568	1584	1600	16
17	1547	1564	1581	1598	1615	1632	1649	1666	1683	1700	17
18	1638	1656	1674	1692	1710	1728	1746	1764	1782	1800	18
19	1729	1748	1767	1786	1805	1824	1843	1862	1881	1900	19
20	1820	1840	1860	1880	1900	1920	1940	1960	1980	2000	20
21	1911	1932	1953	1974	1995	2016	2037	2058	2079	2100	21
22	2002	2024	2046	2068	2090	2112	2134	2156	2178	2200	22
23	2093	2116	2139	2162	2185	2208	2231	2254	2277	2300	23
24	2184	2208	2232	2256	2280	2304	2328	2352	2376	2400	24
25	2275	2300	2325	2350	2375	2400	2425	2450	2475	2500	25
26	2366	2392	2418	2444	2470	2496	2522	2548	2574	2600	26
27	2457	2484	2511	2538	2565	2592	2619	2646	2673	2700	27
28	2548	2576	2604	2632	2660	2688	2716	2744	2772	2800	28
29	2639	2668	2697	2726	2755	2784	2813	2842	2871	2900	29
30	2730	2760	2790	2820	2850	2880	2910	2940	2970	3000	30
31	2821	2852	2883	2914	2945	2976	3007	3038	3069	3100	31
32	2912	2944	2976	3008	3040	3072	3104	3136	3168	3200	32
33	3003	3036	3069	3102	3135	3168	3201	3234	3267	3300	33
34	3094	3128	3162	3196	3230	3264	3298	3332	3366	3400	34
35	3185	3220	3255	3290	3325	3360	3395	3430	3465	3500	35
36	3276	3312	3348	3384	3420	3456	3492	3528	3564	3600	36
37	3367	3404	3441	3478	3515	3552	3589	3626	3663	3700	37
38	3458	3496	3534	3572	3610	3648	3686	3724	3762	3800	38
39	3549	3588	3627	3666	3705	3744	3783	3822	3861	3900	39
40	3640	3680	3720	3760	3800	3840	3880	3920	3960	4000	40
41	3731	3772	3813	3854	3895	3936	3977	4018	4059	4100	41
42	3822	3864	3906	3948	3990	4032	4074	4116	4158	4200	42
43	3913	3956	3999	4042	4085	4128	4171	4214	4257	4300	43
44	4004	4048	4092	4136	4180	4224	4268	4312	4356	4400	44
45	4095	4140	4185	4230	4275	4320	4365	4410	4455	4500	45
46	4186	4232	4278	4324	4370	4416	4462	4508	4554	4600	46
47	4277	4324	4371	4418	4465	4512	4559	4606	4653	4700	47
48	4368	4416	4464	4512	4560	4608	4656	4704	4752	4800	48
49	4459	4508	4557	4606	4655	4704	4753	4802	4851	4900	49
50	4550	4600	4650	4700	4750	4800	4850	4900	4950	5000	50
	91	92	93	94	95	96	97	98	99	100	

	91	92	93	94	95	96	97	98	99	100	
51	4641	4692	4743	4794	4845	4896	4947	4998	5049	5100	**51**
52	4732	4784	4836	4888	4940	4992	5044	5096	5148	5200	**52**
53	4823	4876	4929	4982	5035	5088	5141	5194	5247	5300	**53**
54	4914	4968	5022	5076	5130	5184	5238	5292	5346	5400	**54**
55	5005	5060	5115	5170	5225	5280	5335	5390	5445	5500	**55**
56	5096	5152	5208	5264	5320	5376	5432	5488	5544	5600	**56**
57	5187	5244	5301	5358	5415	5472	5529	5586	5643	5700	**57**
58	5278	5336	5394	5452	5510	5568	5626	5684	5742	5800	**58**
59	5369	5428	5487	5546	5605	5664	5723	5782	5841	5900	**59**
60	5460	5520	5580	5640	5700	5760	5820	5880	5940	6000	**60**
61	5551	5612	5673	5734	5795	5856	5917	5978	6039	6100	**61**
62	5642	5704	5766	5828	5890	5952	6014	6076	6138	6200	**62**
63	5733	5796	5859	5922	5985	6048	6111	6174	6237	6300	**63**
64	5824	5888	5952	6016	6080	6144	6208	6272	6336	6400	**64**
65	5915	5980	6045	6110	6175	6240	6305	6370	6435	6500	**65**
66	6006	6072	6138	6204	6270	6336	6402	6468	6534	6600	**66**
67	6097	6164	6231	6298	6365	6432	6499	6566	6633	6700	**67**
68	6188	6256	6324	6392	6460	6528	6596	6664	6732	6800	**68**
69	6279	6348	6417	6486	6555	6624	6693	6762	6831	6900	**69**
70	6370	6440	6510	6580	6650	6720	6790	6860	6930	7000	**70**
71	6461	6532	6603	6674	6745	6816	6887	6958	7029	7100	**71**
72	6552	6624	6696	6768	6840	6912	6984	7056	7128	7200	**72**
73	6643	6716	6789	6862	6935	7008	7081	7154	7227	7300	**73**
74	6734	6808	6882	6956	7030	7104	7178	7252	7326	7400	**74**
75	6825	6900	6975	7050	7125	7200	7275	7350	7425	7500	**75**
76	6916	6992	7068	7144	7220	7296	7372	7448	7524	7600	**76**
77	7007	7084	7161	7238	7315	7392	7469	7546	7623	7700	**77**
78	7098	7176	7254	7332	7410	7488	7566	7644	7722	7800	**78**
79	7189	7268	7347	7426	7505	7584	7663	7742	7821	7900	**79**
80	7280	7360	7440	7520	7600	7680	7760	7840	7920	8000	**80**
81	7371	7452	7533	7614	7695	7776	7857	7938	8019	8100	**81**
82	7462	7544	7626	7708	7790	7872	7954	8036	8118	8200	**82**
83	7553	7636	7719	7802	7885	7968	8051	8134	8217	8300	**83**
84	7644	7728	7812	7896	7980	8064	8148	8232	8316	8400	**84**
85	7735	7820	7905	7990	8075	8160	8245	8330	8415	8500	**85**
86	7826	7912	7998	8084	8170	8256	8342	8428	8514	8600	**86**
87	7917	8004	8091	8178	8265	8352	8439	8526	8613	8700	**87**
88	8008	8096	8184	8272	8360	8448	8536	8624	8712	8800	**88**
89	8099	8188	8277	8366	8455	8544	8633	8722	8811	8900	**89**
90	8190	8280	8370	8460	8550	8640	8730	8820	8910	9000	**90**
91	8281	8372	8463	8554	8645	8736	8827	8918	9009	9100	**91**
92	8372	8464	8556	8648	8740	8832	8924	9016	9108	9200	**92**
93	8463	8556	8649	8742	8835	8928	9021	9114	9207	9300	**93**
94	8554	8648	8742	8836	8930	9024	9118	9212	9306	9400	**94**
95	8645	8740	8835	8930	9025	9120	9215	9310	9405	9500	**95**
96	8736	8832	8928	9024	9120	9216	9312	9408	9504	9600	**96**
97	8827	8924	9021	9118	9215	9312	9409	9506	9603	9700	**97**
98	8918	9016	9114	9212	9310	9408	9506	9604	9702	9800	**98**
99	9009	9108	9207	9306	9405	9504	9603	9702	9801	9900	**99**
100	9100	9200	9300	9400	9500	9600	9700	9800	9900	10000	**100**
	91	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**	**100**	

TABLE 61

A MULTIPLICATION TABLE

Giving the Products of $1^2, 2^2 \ldots (12)^2$ Times 1, 2, 3 . . . 100 and the Products of $(13)^2 \ldots (21)^2$ Times 1, 2, 3 . . . 50.

Products (1 to 50) × 2^2, 3^2, 4^2, etc.

	2^2	3^2	4^2	5^2	6^2	7^2	8^2	9^2	$(11)^2$	$(12)^2$	
1	4	9	16	25	36	49	64	81	121	144	1
2	8	18	32	50	72	98	128	162	242	288	2
3	12	27	48	75	108	147	192	243	363	432	3
4	16	36	64	100	144	196	256	324	484	576	4
5	20	45	80	125	180	245	320	405	605	720	5
6	24	54	96	150	216	294	384	486	726	864	6
7	28	63	112	175	252	343	448	567	847	1008	7
8	32	72	128	200	288	392	512	648	968	1152	8
9	36	81	144	225	324	441	576	729	1089	1296	9
10	40	90	160	250	360	490	640	810	1210	1440	10
11	44	99	176	275	396	539	704	891	1331	1584	11
12	48	108	192	300	432	588	768	972	1452	1728	12
13	52	117	208	325	468	637	832	1053	1573	1872	13
14	56	126	224	350	504	686	896	1134	1694	2016	14
15	60	135	240	375	540	735	960	1215	1815	2160	15
16	64	144	256	400	576	784	1024	1296	1936	2304	16
17	68	153	272	425	612	833	1088	1377	2057	2448	17
18	72	162	288	450	648	882	1152	1458	2178	2592	18
19	76	171	304	475	684	931	1216	1539	2299	2736	19
20	80	180	320	500	720	980	1280	1620	2420	2880	20
21	84	189	336	525	756	1029	1344	1701	2541	3024	21
22	88	198	352	550	792	1078	1408	1782	2662	3168	22
23	92	207	368	575	828	1127	1472	1863	2783	3312	23
24	96	216	384	600	864	1176	1536	1944	2904	3456	24
25	100	225	400	625	900	1225	1600	2025	3025	3600	25
26	104	234	416	650	936	1274	1664	2106	3146	3744	26
27	108	243	432	675	972	1323	1728	2187	3267	3888	27
28	112	252	448	700	1008	1372	1792	2268	3388	4032	28
29	116	261	464	725	1044	1421	1856	2349	3509	4176	29
30	120	270	480	750	1080	1470	1920	2430	3630	4320	30
31	124	279	496	775	1116	1519	1984	2511	3751	4464	31
32	128	288	512	800	1152	1568	2048	2592	3872	4608	32
33	132	297	528	825	1188	1617	2112	2673	3993	4752	33
34	136	306	544	850	1224	1666	2176	2754	4114	4896	34
35	140	315	560	875	1260	1715	2240	2835	4235	5040	35
36	144	324	576	900	1296	1764	2304	2916	4356	5184	36
37	148	333	592	925	1332	1813	2368	2997	4477	5328	37
38	152	342	608	950	1368	1862	2432	3078	4598	5472	38
39	156	351	624	975	1404	1911	2496	3159	4719	5616	39
40	160	360	640	1000	1440	1960	2560	3240	4840	5760	40
41	164	369	656	1025	1476	2009	2624	3321	4961	5904	41
42	168	378	672	1050	1512	2058	2688	3402	5082	6048	42
43	172	387	688	1075	1548	2107	2752	3483	5203	6192	43
44	176	396	704	1100	1584	2156	2816	3564	5324	6336	44
45	180	405	720	1125	1620	2205	2880	3645	5445	6480	45
46	184	414	736	1150	1656	2254	2944	3726	5566	6624	46
47	188	423	752	1175	1692	2303	3008	3807	5687	6768	47
48	192	432	768	1200	1728	2352	3072	3888	5808	6912	48
49	196	441	784	1225	1764	2401	3136	3969	5929	7056	49
50	200	450	800	1250	1800	2450	3200	4050	6050	7200	50
	2^2	3^2	4^2	5^2	6^2	7^2	8^2	9^2	$(11)^2$	$(12)^2$	

Products (51 to 100) × 2^2, 3^2, 4^2, etc.

	2^2	3^2	4^2	5^2	6^2	7^2	8^2	9^2	$(11)^2$	$(12)^2$	
51	204	459	816	1275	1836	2499	3264	4131	6171	7344	**51**
52	208	468	832	1300	1872	2548	3328	4212	6292	7488	**52**
53	212	477	848	1325	1908	2597	3392	4293	6413	7632	**53**
54	216	486	864	1350	1944	2646	3456	4374	6534	7776	**54**
55	220	495	880	1375	1980	2695	3520	4455	6655	7920	**55**
56	224	504	896	1400	2016	2744	3584	4536	6776	8064	**56**
57	228	513	912	1425	2052	2793	3648	4617	6897	8208	**57**
58	232	522	928	1450	2088	2842	3712	4698	7018	8352	**58**
59	236	531	944	1475	2124	2891	3776	4779	7139	8496	**59**
60	240	540	960	1500	2160	2940	3840	4860	7260	8640	**60**
61	244	549	976	1525	2196	2989	3904	4941	7381	8784	**61**
62	248	558	992	1550	2232	3038	3968	5022	7502	8928	**62**
63	252	567	1008	1575	2268	3087	4032	5103	7623	9072	**63**
64	256	576	1024	1600	2304	3136	4096	5184	7744	9216	**64**
65	260	585	1040	1625	2340	3185	4160	5265	7865	9360	**65**
66	264	594	1056	1650	2376	3234	4224	5346	7986	9504	**66**
67	268	603	1072	1675	2412	3283	4288	5427	8107	9648	**67**
68	272	612	1088	1700	2448	3332	4352	5508	8228	9792	**68**
69	276	621	1104	1725	2484	3381	4416	5589	8349	9936	**69**
70	280	630	1120	1750	2520	3430	4480	5670	8470	10080	**70**
71	284	639	1136	1775	2556	3479	4544	5751	8591	10224	**71**
72	288	648	1152	1800	2592	3528	4608	5832	8712	10368	**72**
73	292	657	1168	1825	2628	3577	4672	5913	8833	10512	**73**
74	296	666	1184	1850	2664	3626	4736	5994	8954	10656	**74**
75	300	675	1200	1875	2700	3675	4800	6075	9075	10800	**75**
76	304	684	1216	1900	2736	3724	4864	6156	9196	10944	**76**
77	308	693	1232	1925	2772	3773	4928	6237	9317	11088	**77**
78	312	702	1248	1950	2808	3822	4992	6318	9438	11232	**78**
79	316	711	1264	1975	2844	3871	5056	6399	9559	11376	**79**
80	320	720	1280	2000	2880	3920	5120	6480	9680	11520	**80**
81	324	729	1296	2025	2916	3969	5184	6561	9801	11664	**81**
82	328	738	1312	2050	2952	4018	5248	6642	9922	11808	**82**
83	332	747	1328	2075	2988	4067	5312	6723	10043	11952	**83**
84	336	756	1344	2100	3024	4116	5376	6804	10164	12096	**84**
85	340	765	1360	2125	3060	4165	5440	6885	10285	12240	**85**
86	344	774	1376	2150	3096	4214	5504	6966	10406	12384	**86**
87	348	783	1392	2175	3132	4263	5568	7047	10527	12528	**87**
88	352	792	1408	2200	3168	4312	5632	7128	10648	12672	**88**
89	356	801	1424	2225	3204	4361	5696	7209	10769	12816	**89**
90	360	810	1440	2250	3240	4410	5760	7290	10890	12960	**90**
91	364	819	1456	2275	3276	4459	5824	7371	11011	13104	**91**
92	368	828	1472	2300	3312	4508	5888	7452	11132	13248	**92**
93	372	837	1488	2325	3348	4557	5952	7533	11253	13392	**93**
94	376	846	1504	2350	3384	4606	6016	7614	11374	13536	**94**
95	380	855	1520	2375	3420	4655	6080	7695	11495	13680	**95**
96	384	864	1536	2400	3456	4704	6144	7776	11616	13824	**96**
97	388	873	1552	2425	3492	4753	6208	7857	11727	13968	**97**
98	392	882	1568	2450	3528	4802	6272	7938	11858	14112	**98**
99	396	891	1584	2475	3564	4851	6336	8019	11979	14256	**99**
100	400	900	1600	2500	3600	4900	6400	8100	12100	14440	**100**
	2^2	3^2	4^2	5^2	6^2	7^2	8^2	9^2	$(11)^2$	$(12)^2$	

Products (1 to 50) × $(13)^2$, $(14)^2$, $(15)^2$, etc

	$(13)^2$	$(14)^2$	$(15)^2$	$(16)^2$	$(17)^2$	$(18)^2$	$(19)^2$	$(21)^2$	
1	169	196	225	256	289	324	361	441	1
2	338	392	450	512	578	648	722	882	2
3	507	588	675	768	867	972	1083	1323	3
4	676	784	900	1024	1156	1296	1444	1764	4
5	845	980	1125	1280	1445	1620	1805	2205	5
6	1014	1176	1350	1536	1734	1944	2166	2646	6
7	1183	1372	1575	1792	2023	2268	2527	3087	7
8	1352	1568	1800	2048	2312	2592	2888	3528	8
9	1521	1764	2025	2304	2601	2916	3249	3969	9
10	1690	1960	2250	2560	2890	3240	3610	4410	10
11	1859	2156	2475	2816	3179	3564	3971	4851	11
12	2028	2352	2700	3072	3468	3888	4332	5292	12
13	2197	2548	2925	3328	3757	4212	4693	5733	13
14	2366	2744	3150	3584	4046	4536	5054	6174	14
15	2535	2940	3375	3840	4335	4860	5415	6615	15
16	2704	3136	3600	4096	4624	5184	5776	7056	16
17	2873	3332	3825	4352	4913	5508	6137	7497	17
18	3042	3528	4050	4608	5202	5832	6498	7938	18
19	3211	3724	4275	4864	5491	6156	6859	8379	19
20	3380	3920	4500	5120	5780	6480	7220	8820	20
21	3549	4116	4725	5376	6069	6804	7581	9261	21
22	3718	4312	4950	5632	6358	7128	7942	9702	22
23	3887	4508	5175	5888	6647	7452	8303	10143	23
24	4056	4704	5400	6144	6936	7776	8664	10584	24
25	4225	4900	5625	6400	7225	8100	9025	11025	25
26	4394	5096	5850	6656	7514	8424	9386	11466	26
27	4563	5292	6075	6912	7803	8748	9747	11907	27
28	4732	5488	6300	7168	8092	9072	10108	12348	28
29	4901	5684	6525	7424	8381	9396	10469	12789	29
30	5070	5880	6750	7680	8670	9720	10830	13230	30
31	5239	6076	6975	7936	8959	10044	11191	13671	31
32	5408	6272	7200	8192	9248	10368	11552	14112	32
33	5577	6468	7425	8448	9537	10692	11913	14553	33
34	5746	6664	7650	8704	9826	11016	12274	14994	34
35	5915	6860	7875	8960	10115	11340	12635	15435	35
36	6084	7056	8100	9216	10404	11664	12996	15876	36
37	6253	7252	8325	9472	10693	11988	13357	16317	37
38	6422	7448	8550	9728	10982	12312	13718	16758	38
39	6591	7644	8775	9984	11271	12636	14079	17199	39
40	6760	7840	9000	10240	11560	12960	14440	17640	40
41	6929	8036	9225	10496	11849	13284	14801	18081	41
42	7098	8232	9450	10752	12138	13608	15162	18522	42
43	7267	8428	9675	11008	12427	13932	15523	18963	43
44	7436	8624	9900	11264	12716	14256	15884	19404	44
45	7605	8820	10125	11520	13005	14580	16245	19845	45
46	7774	9016	10350	11776	13294	14904	16606	20286	46
47	7943	9212	10575	12032	13583	15228	16967	20727	47
48	8112	9408	10800	12288	13872	15552	17328	21168	48
49	8281	9604	11025	12544	14161	15876	17689	21609	49
50	8450	9800	11250	12800	14450	16200	18050	22050	50
	$(13)^2$	$(14)^2$	$(15)^2$	$(16)^2$	$(17)^2$	$(18)^2$	$(19)^2$	$(21)^2$	

TABLE 62

A TABLE OF THE SQUARES AND SQUARE ROOTS OF THE NUMBERS FROM 1 TO 1000.

THIS table is a modification of the first part of Barlow's Tables. The advantage of this abridged table beyond its more convenient size, is that through the omission of cubes, cube roots and reciprocals, the table allows more rapid use and causes much less strain on the eyes. The latter result is furthered by giving square roots only to the third decimal instead of to the seventh.

Num.	Square.	Squ. Root.	Num.	Square.	Squ. Root.
1	1	1.000	51	26 01	7.141
2	4	1.414	52	27 04	7.211
3	9	1.732	53	28 09	7.280
4	16	2.000	54	29 16	7.348
5	25	2.236	55	30 25	7.416
6	36	2.449	56	31 36	7.483
7	49	2.646	57	32 49	7.550
8	64	2.828	58	33 64	7.616
9	81	3.000	59	34 81	7.681
10	1 00	3.162	60	36 00	7.746
11	1 21	3.317	61	37 21	7.810
12	1 44	3.464	62	38 44	7.874
13	1 69	3.606	63	39 69	7.937
14	1 96	3.742	64	40 96	8.000
15	2 25	3.873	65	42 25	8.062
16	2 56	4.000	66	43 56	8.124
17	2 89	4.123	67	44 89	8.185
18	3 24	4.243	68	46 24	8.246
19	3 61	4.359	69	47 61	8.307
20	4 00	4.472	70	49 00	8.367
21	4 41	4.583	71	50 41	8.426
22	4 84	4.690	72	51 84	8.485
23	5 29	4.796	73	53 29	8.544
24	5 76	4.899	74	54 76	8.602
25	6 25	5.000	75	56 25	8.660
26	6 76	5.099	76	57 76	8.718
27	7 29	5.196	77	59 29	8.775
28	7 84	5.292	78	60 84	8.832
29	8 41	5.385	79	62 41	8.888
30	9 00	5.477	80	64 00	8.944
31	9 61	5.568	81	65 61	9.000
32	10 24	5.657	82	67 24	9.055
83	10 89	5.745	83	68 89	9.110
34	11 56	5.831	84	70 56	9.165
35	12 25	5.916	85	72 25	9.220
36	12 96	6.000	86	73 96	9.274
37	13 69	6.083	87	75 69	9.327
38	14 44	6.164	88	77 44	9.381
39	15 21	6.245	89	79 21	9.434
40	16 00	6.325	90	81 00	9.487
41	16 81	6.403	91	82 81	9.539
42	17 64	6.481	92	84 64	9.592
43	18 49	6.557	93	86 49	9.644
44	19 36	6.633	94	88 36	9.695
45	20 25	6.708	95	90 25	9.747
46	21 16	6.782	96	92 16	9.798
47	22 09	6.856	97	94 09	9.849
48	23 04	6.928	98	96 04	9.899
49	24 01	7.000	99	98 01	9.950
50	25 00	7.071	100	1 00 00	10.000

Num.	Square.	Squ. Root.	Num.	Square.	Squ. Root.
101	1 02 01	10.050	151	2 28 01	12.288
102	1 04 04	10.100	152	2 31 04	12.329
103	1 06 09	10.149	153	2 34 09	12.369
104	1 08 16	10.198	154	2 37 16	12.410
105	1 10 25	10.247	155	2 40 25	12.450
106	1 12 36	10.296	156	2 43 36	12.490
107	1 14 49	10.344	157	2 46 49	12.530
108	1 16 64	10.392	158	2 49 64	12.570
109	1 18 81	10.440	159	2 52 81	12.610
110	1 21 00	10.488	160	2 56 00	12.649
111	1 23 21	10.536	161	2 59 21	12.689
112	1 25 44	10.583	162	2 62 44	12.728
113	1 27 69	10.630	163	2 65 69	12.767
114	1 29 96	10.677	164	2 68 96	12.806
115	1 32 25	10.724	165	2 72 25	12.845
116	1 34 56	10.770	166	2 75 56	12.884
117	1 36 89	10.817	167	2 78 89	12.923
118	1 39 24	10.863	168	2 82 24	12.961
119	1 41 61	10.909	169	2 85 61	13.000
120	1 44 00	10.954	170	2 89 00	13.038
121	1 46 41	11.000	171	2 92 41	13.077
122	1 48 84	11.045	172	2 95 84	13.115
123	1 51 29	11.091	173	2 99 29	13.153
124	1 53 76	11.136	174	3 02 76	13.191
125	1 56 25	11.180	175	3 06 25	13.229
126	1 58 76	11.225	176	3 09 76	13.266
127	1 61 29	11.269	177	3 13 29	13.304
128	1 63 84	11.314	178	3 16 84	13.342
129	1 66 41	11.358	179	3 20 41	13.379
130	1 69 00	11.402	180	3 24 00	13.416
131	1 71 61	11.446	181	3 27 61	13.454
132	1 74 24	11.489	182	3 31 24	13.491
133	1 76 89	11.533	183	3 34 89	13.528
134	1 79 56	11.576	184	3 38 56	13.565
135	1 82 25	11.619	185	3 42 25	13.601
136	1 84 96	11.662	186	3 45 96	13.638
137	1 87 69	11.705	187	3 49 69	13.675
138	1 90 44	11.747	188	3 53 44	13.711
139	1 93 21	11.790	189	3 57 21	13.748
140	1 96 00	11.832	190	3 61 00	13.784
141	1 98 81	11.874	191	3 64 81	13.820
142	2 01 64	11.916	192	3 68 64	13.856
143	2 04 49	11.958	193	3 72 49	13.892
144	2 07 36	12.000	194	3 76 36	13.928
145	2 10 25	12.042	195	3 80 25	13.964
146	2 13 16	12.083	196	3 84 16	14.000
147	2 16 09	12.124	197	3 88 09	14.036
148	2 19 04	12.166	198	3 92 04	14.071
149	2 22 01	12.207	199	3 96 01	14.107
150	2 25 00	12.247	200	4 00 00	14.142

Nnm.	Square.	Squ. Root.	Num.	Square.	Squ. Root.
201	4 04 01	14.177	251	6 30 01	15.843
202	4 08 04	14.213	252	6 35 04	15.875
203	4 12 09	14.248	253	6 40 09	15.906
204	4 16 16	14.283	254	6 45 16	15.937
205	4 20 25	14.318	255	6 50 25	15.969
206	4 24 36	14.353	256	6 55 36	16.000
207	4 28 49	14.387	257	6 60 49	16.031
208	4 32 64	14.422	258	6 65 64	16.062
209	4 36 81	14.457	259	6 70 81	16.093
210	4 41 00	14.491	260	6 76 00	16.125
211	4 45 21	14.526	261	6 81 21	16.155
212	4 49 44	14.560	262	6 86 44	16.186
213	4 53 69	14.595	263	6 91 69	16.217
214	4 57 96	14.629	264	6 96 96	16.248
215	4 62 25	14.663	265	7 02 25	16.279
216	4 66 56	14.697	266	7 07 56	16.310
217	4 70 89	14.731	267	7 12 89	16.340
218	4 75 24	14.765	268	7 18 24	16.371
219	4 79 61	14.799	269	7 23 61	16.401
220	4 84 00	14.832	270	7 29 00	16.432
221	4 88 41	14.866	271	7 34 41	16.462
222	4 92 84	14.900	272	7 39 84	16.492
223	4 97 29	14.933	273	7 45 29	16.523
224	5 01 76	14.967	274	7 50 76	16.553
225	5 06 25	15.000	275	7 56 25	16.583
226	5 10 76	15.033	276	7 61 76	16.613
227	5 15 29	15.067	277	7 67 29	16.643
228	5 19 84	15.100	278	7 72 84	16.673
229	5 24 41	15.133	279	7 78 41	16.703
230	5 29 00	15.166	280	7 84 00	16.733
231	5 33 61	15.199	281	7 89 61	16.763
232	5 38 24	15.232	282	7 95 24	16.793
233	5 42 89	15.264	283	8 00 89	16.823
234	5 47 56	15.297	284	8 06 56	16.852
235	5 52 25	15.330	285	8 12 25	16.882
236	5 56 96	15.362	286	8 17 96	16.912
237	5 61 69	15.395	287	8 23 69	16.941
238	5 66 44	15.427	288	8 29 44	16.971
239	5 71 21	15.460	289	8 35 21	17.000
240	5 76 00	15.492	290	8 41 00	17.029
241	5 80 81	15.524	291	8 46 81	17.059
242	5 85 64	15.556	292	8 52 64	17.088
243	5 90 49	15.588	293	8 58 49	17.117
244	5 95 36	15.620	294	8 64 36	17.146
245	6 00 25	15.652	295	8 70 25	17.176
246	6 05 16	15.684	296	8 76 16	17.205
247	6 10 09	15.716	297	8 82 09	17.234
248	6 15 04	15.748	298	8 88 04	17.263
249	6 20 01	15.780	299	8 94 01	17.292
250	6 25 00	15.811	300	9 00 00	17.321

Num.	Square.	Squ. Root.	Num.	Square.	Squ. Root.
301	9 06 01	17.349	351	12 32 01	18.735
302	9 12 04	17.378	352	12 39 04	18.762
303	9 18 09	17.407	353	12 46 09	18.788
304	9 24 16	17.436	354	12 53 16	18.815
305	9 30 25	17.464	355	12 60 25	18.841
306	9 36 36	17.493	356	12 67 36	18.868
307	9 42 49	17.521	357	12 74 49	18.894
308	9 48 64	17.550	358	12 81 64	18.921
309	9 54 81	17.578	359	12 88 81	18.947
310	9 61 00	17.607	360	12 96 00	18.974
311	9 67 21	17.635	361	13 03 21	19.000
312	9 73 44	17.664	362	13 10 44	19.026
313	9 79 69	17.692	363	13 17 69	19.053
314	9 85 96	17 720	364	13 24 96	19.079
315	9 92 25	17.748	365	13 32 25	19.105
316	9 98 56	17.776	366	13 39 56	19.131
317	10 04 89	17.804	367	13 46 89	19.157
318	10 11 24	17.833	368	13 54 24	19.183
319	10 17 61	17.861	369	13 61 61	19.209
320	10 24 00	17.889	370	13 69 00	19.235
321	10 30 41	17.916	371	13 76 41	19.261
322	10 36 84	17.944	372	13 83 84	19.287
323	10 43 29	17.972	373	13 91 29	19.313
324	10 49 76	18.000	374	13 98 76	19.339
325	10 56 25	18.028	375	14 06 25	19.365
326	10 62 76	18.055	376	14 13 76	19.391
327	10 69 29	18.083	377	14 21 29	19.416
328	10 75 84	18.111	378	14 28 84	19.442
329	10 82 41	18.138	379	14 36 41	19.468
330	10 89 00	18.166	380	14 44 00	19.494
331	10 95 61	18.193	381	14 51 61	19.519
332	11 02 24	18.221	382	14 59 24	19.545
333	11 08 89	18.248	383	14 66 89	19.570
334	11 15 56	18.276	384	14 74 56	19.596
335	11 22 25	18.303	385	14 82 25	19.621
336	11 28 96	18.330	386	14 89 96	19.647
337	11 35 69	18.358	387	14 97 69	19.672
338	11 42 44	18.385	388	15 05 44	19.698
339	11 49 21	18.412	389	15 13 21	19.723
340	11 56 00	18.439	390	15 21 00	19.748
341	11 62 81	18.466	391	15 28 81	19.774
342	11 69 64	18.493	392	15 36 64	19.799
343	11 76 49	18.520	393	15 44 49	19.824
344	11 83 36	18.547	394	15 52 36	19 849
345	11 90 25	18.574	395	15 60 25	19.875
346	11 97 16	18.601	396	15 68 16	19.900
347	12 04 09	18.628	397	15 76 09	19.925
348	12 11 04	18.655	398	15 84 04	19 950
349	12 18 01	18.682	399	15 92 01	19.975
350	12 25 00	18.708	400	16 00 00	20.000

Num.	Square.	Squ. Root.
401	16 08 01	20.025
402	16 16 04	20.050
403	16 24 09	20.075
404	16 32 16	20.100
405	16 40 25	20.125
406	16 48 36	20.149
407	16 56 49	20.174
408	16 64 64	20.199
409	16 72 81	20.224
410	16 81 00	20.248
411	16 89 21	20.273
412	16 97 44	20.298
413	17 05 69	20.322
414	17 13 96	20.347
415	17 22 25	20.372
416	17 30 56	20.396
417	17 38 89	20.421
418	17 47 24	20.445
419	17 55 61	20.469
420	17 64 00	20.494
421	17 72 41	20.518
422	17 80 84	20.543
423	17 89 29	20.567
424	17 97 76	20.591
425	18 06 25	20.616
426	18 14 76	20.640
427	18 23 29	20.664
428	18 31 84	20.688
429	18 40 41	20.712
430	18 49 00	20.736
431	18 57 61	20.761
432	18 66 24	20.785
433	18 74 89	20.809
434	18 83 56	20.833
435	18 92 25	20.857
436	19 00 96	20 881
437	19 09 69	20.905
438	19 18 44	20.928
439	19 27 21	20.952
440	19 36 00	20.976
441	19 44 81	21.000
442	19 53 64	21.024
443	19 62 49	21.048
444	19 71 36	21.071
445	19 80 25	21.095
446	19 89 16	21.119
447	19 98 09	21.142
448	20 07 04	21.166
449	20 16 01	21.190
450	20 25 00	21 213
451	20 34 01	21.237
452	20 43 04	21.260
453	20 52 09	21.284
454	20 61 16	21.307
455	20 70 25	21.331
456	20 79 36	21.354
457	20 88 49	21.378
458	20 97 64	21 401
459	21 06 81	21.424
460	21 16 00	21.448
461	21 25 21	21.471
462	21 34 44	21.494
463	21 43 69	21.517
464	21 52 96	21.541
465	21 62 25	21.564
466	21 71 56	21.587
467	21 80 89	21.610
468	21 90 24	21.633
469	21 99 61	21.656
470	22 09 00	21.679
471	22 18 41	21.703
472	22 27 84	21.726
473	22 37 29	21.749
474	22 46 76	21.772
475	22 56 25	21.794
476	22 65 76	21.817
477	22 75 29	21.840
478	22 84 84	21.863
479	22 94 41	21.886
480	23 04 00	21.909
481	23 13 61	21.932
482	23 23 24	21.954
483	23 32 89	21.977
484	23 42 56	22.000
485	23 52 25	22.023
486	23 61 96	22.045
487	23 71 69	22.068
488	23 81 44	22.091
489	23 91 21	22.113
490	24 01 00	22.136
491	24 10 81	22.159
492	24 20 64	22.181
493	24 30 49	22.204
494	24 40 36	22.226
495	24 50 25	22.249
496	24 60 16	22.271
497	24 70 09	22.293
498	24 80 04	22.316
499	24 90 01	22.338
500	25 00 00	22.361

Num.	Square.	Squ. Root.	Num.	Square.	Squ. Root.
501	25 10 01	22.383	551	30 36 01	23.473
502	25 20 04	22.405	552	30 47 04	23.495
503	25 30 09	22.428	553	30 58 09	23.516
504	25 40 16	22.450	554	80 69 16	23.537
505	25 50 25	22.472	555	30 80 25	23.558
506	25 60 36	22.494	556	30 91 36	23.580
507	25 70 49	22.517	557	31 02 49	23.601
508	25 80 64	22.539	558	31 13 64	23.622
509	25 90 81	22.561	559	31 24 81	23.643
510	26 01 00	22.583	560	31 36 00	23.664
511	26 11 21	22.605	561	31 47 21	23.685
512	26 21 44	22.627	562	31 58 44	23.707
513	26 31 69	22.650	563	31 69 69	23.728
514	26 41 96	22.672	564	31 80 96	23.749
515	26 52 25	22.694	565	31 92 25	23.770
516	26 62 56	22.716	566	32 03 56	23.791
517	26 72 89	22.738	567	32 14 89	23.812
518	26 83 24	22.760	568	32 26 24	23.833
519	26 93 61	22.782	569	32 37 61	23.854
520	27 04 00	22.804	570	32 49 00	23.875
521	27 14 41	22.825	571	32 60 41	23.896
522	27 24 84	22.847	572	32 71 84	23.917
523	27 35 29	22.869	573	32 83 29	23.937
524	27 45 76	22.891	574	32 94 76	23.958
525	27 56 25	22.913	575	33 06 25	23.979
526	27 66 76	22.935	576	33 17 76	24.000
527	27 77 29	22.956	577	33 29 29	24.021
528	27 87 84	22.978	578	33 40 84	24.042
529	27 98 41	23.000	579	33 52 41	24.062
530	28 09 00	23.022	580	33 64 00	24.083
531	28 19 61	23.043	581	33 75 61	24.104
532	28 30 24	23.065	582	33 87 24	24.125
533	28 40 89	23.087	583	33 98 89	24.145
534	28 51 56	23.108	584	34 10 56	24.166
535	28 62 25	23.130	585	34 22 25	24.187
536	28 72 96	23.152	586	34 33 96	24.207
537	28 83 69	23.173	587	34 45 69	24.228
538	28 94 44	23.195	588	34 57 44	24.249
539	29 05 21	23.216	5/ 589	34 69 21	24.269
540	29 16 00	23.238	590	34 81 00	24.290
541	29 26 81	23.259	591	34 92 81	24.310
542	29 37 64	23.281	592	35 04 64	24.331
543	29 48 49	23.302	593	35 16 49	24.352
544	29 59 36	23.324	594	35 28 36	24.372
545	29 70 25	23.345	595	35 40 25	24.393
546	29 81 16	23.367	596	35 52 16	24.413
547	29 92 09	23.388	597	35 64 09	24.434
548	30 03 04	23.409	598	35 76 04	24.454
549	30 14 01	23.431	599	35 88 01	24.474
550	30 25 00	23.452	600	36 00 00	24.495

Num.	Square.	Squ. Root.
601	36 12 01	24.515
602	36 24 04	24.536
603	36 36 09	24.556
604	36 48 16	24.576
605	36 60 25	24.597
606	36 72 36	24.617
607	36 84 49	24.637
608	36 96 64	24.658
609	37 08 81	24.678
610	37 21 00	24.698
611	37 33 21	24.718
612	37 45 44	24.739
613	37 57 69	24.759
614	37 69 96	24.779
615	37 82 25	24.799
616	37 94 56	24.819
617	38 06 89	24.839
618	38 19 24	24.860
619	38 31 61	24.880
620	38 44 00	24.900
621	38 56 41	24.920
622	38 68 84	24.940
623	38 81 29	24.960
624	38 93 76	24.980
625	39 06 25	25.000
626	39 18 76	25.020
627	39 31 29	25.040
628	39 43 84	25.060
629	39 56 41	25.080
630	39 69 00	25.100
631	39 81 61	25.120
632	39 94 24	25.140
633	40 06 89	25.159
634	40 19 56	35.179
635	40 32 25	25.199
636	40 44 96	25.219
637	40 57 69	25.239
638	40 70 44	25.259
639	40 83 21	25.278
640	40 96 00	25.298
641	41 08 81	25.318
642	41 21 64	25.338
643	41 34 49	25.357
644	41 47 36	25.377
645	41 60 25	25.397
646	41 73 16	25.417
647	41 86 09	25.436
648	41 99 04	25.456
649	42 12 01	25.475
650	42 25 00	25.495
651	42 38 01	25.515
652	42 51 04	25.534
653	42 64 09	25.554
654	42 77 16	25.573
655	42 90 25	25.593
656	43 03 36	25.612
657	43 16 49	25.632
658	43 29 64	25.652
659	43 42 81	25.671
660	43 56 00	25.690
661	43 69 21	25.710
662	43 82 44	25.729
663	43 95 69	25.749
664	44 08 96	25.768
665	44 22 25	25.788
666	44 35 56	25.807
667	44 48 89	25.826
668	44 62 24	25.846
669	44 75 61	25.865
670	44 89 00	25.884
671	45 02 41	25.904
672	45 15 84	25.923
673	45 29 29	25.942
674	45 42 76	25.962
675	45 56 25	25.981
676	45 69 76	26.000
677	45 83 29	26.019
678	45 96 84	26.038
679	46 10 41	26.058
680	46 24 00	26.077
681	46 37 61	26.096
682	46 51 24	26.115
683	46 64 89	26.134
684	46 78 56	26.153
685	46 92 25	26.173
686	47 05 96	26.192
687	47 19 69	26.211
688	47 33 44	26 230
689	47 47 21	26.249
690	47 61 00	26.268
691	47 74 81	26.287
692	47 88 64	26.306
693	48 02 49	26.325
694	48 16 36	26.344
695	48 30 25	26.363
696	48 44 16	26.382
697	48 58 09	26.401
698	48 72 04	26.420
699	48 86 01	26.439
700	49 00 00	26.458

Num.	Square.	Squ. Root.	Num.	Square.	Squ. Root.
701	49 14 01	26.476	751	56 40 01	27 404
702	49 28 04	26.495	752	56 55 04	27.423
703	49 42 09	26.514	753	56 70 09	27.441
704	49 56 16	26.533	754	56 85 16	27.459
705	49 70 25	26.552	755	57 00 25	27.477
706	49 84 36	26.571	756	57 15 36	27.495
707	49 98 49	26.589	757	57 30 49	27.514
708	50 12 64	26.608	758	57 45 64	27.532
709	50 26 81	26.627	759	57 60 81	27.550
710	50 41 00	26.646	760	57 76 00	27.568
711	50 55 21	26.665	761	57 91 21	27.586
712	50 69 44	26.683	762	58 06 44	27.604
713	50 83 69	26.702	763	58 21 69	27.622
714	50 97 96	26.721	764	58 36 96	27.641
715	51 12 25	26.739	765	58 52 25	27.659
716	51 26 56	26.758	766	58 67 56	27.677
717	51 40 89	26.777	767	58 82 89	27.695
718	51 55 24	26.796	768	58 98 24	27.713
719	51 69 61	26.814	769	59 13 61	27.731
720	51 84 00	26.833	770	59 29 00	27.749
721	51 98 41	26.851	771	59 44 41	27.767
722	52 12 84	26.870	772	59 59 84	27.785
723	52 27 29	26.889	773	59 75 29	27.803
724	52 41 76	26 907	774	59 90 76	27.821
725	52 56 25	26.926	775	60 06 25	27.839
726	52 70 76	26.944	776	60 21 76	27.857
727	52 85 29	26.963	777	60 37 29	27.875
728	52 99 84	26.981	778	60 52 84	27.893
729	53 14 41	27.000	779	60 68 41	27.911
730	53 29 00	27.019	780	60 84 00	27.928
731	53 43 61	27.037	781	60 99 61	27.946
732	53 58 24	27.055	782	61 15 24	27.964
733	53 72 89	27.074	783	61 30 89	27.982
734	53 87 56	27.092	784	61 46 56	28.000
735	54 02 25	27.111	785	61 62 25	28.018
736	54 16 96	27.129	786	61 77 96	28.036
737	54 31 69	27.148	787	61 93 69	28.054
738	54 46 44	27.166	788	62 09 44	28.071
739	54 61 21	27.185	789	62 25 21	28.089
740	54 76 00	27.203	790	62 41 00	28.107
741	54 90 81	27.221	791	62 56 81	28.125
742	55 05 64	27.240	792	62 72 64	28.142
743	55 20 49	27.258	793	62 88 49	28.160
744	55 35 36	27.276	794	63 04 36	28.178
745	55 50 25	27.295	795	63 20 25	28.196
746	55 65 16	27.313	796	63 36 16	28.213
747	55 80 09	27.331	797	63 52 09	28.231
748	55 95 04	27.350	798	63 68 04	28.249
749	56 10 01	27.368	799	63 84 01	28.267
750	56 25 00	27.386	800	64 00 00	28.284

Num.	Square.	Squ. Root.	Num.	Square.	Squ. Root.
801	64 16 01	28.302	851	72 42 01	29.172
802	64 32 04	28.320	852	72 59 04	29.189
803	64 48 09	28.337	853	72 76 09	29.206
804	64 64 16	28.355	854	72 93 16	29.223
805	64 80 25	28.373	855	73 10 25	29.240
806	64 96 36	28.390	856	73 27 36	29.257
807	65 12 49	28.408	857	73 44 49	29.275
808	65 28 64	28.425	858	73 61 64	29.292
809	65 44 81	28.443	859	73 78 81	29.309
810	65 61 00	28.460	860	73 96 00	29.326
811	65 77 21	28.478	861	74 13 21	29.343
812	65 93 44	28.496	862	74 30 44	29 360
813	66 09 69	28.513	863	74 47 69	29.377
814	66 25 96	28.531	864	74 64 96	29.394
815	66 42 25	28.548	865	74 82 25	29.411
816	66 58 56	28 566	866	74 99 56	29.428
817	66 74 89	28.583	867	75 16 89	29.445
818	66 91 24	28.601	868	75 34 24	29.462
819	67 07 61	28.618	869	75 51 61	29.479
820	67 24 00	28.636	870	75 69 00	29.496
821	67 40 41	28.653	871	75 86 41	29.513
822	67 56 84	28.671	872	76 03 84	29.530
823	67 73 29	28.688	873	76 21 29	29.547
824	67 89 76	28.705	874	76 38 76	29.563
825	68 06 25	28.723	875	76 56 25	29.580
826	68 22 76	28.740	876	76 73 76	29.597
827	68 39 29	28.758	877	76 91 29	29.614
828	68 55 84	28.775	878	77 08 84	29.631
829	68 72 41	28.792	879	77 26 41	29.648
830	68 89 00	28.810	880	77 44 00	29.665
831	69 05 61	28.827	881	77 61 61	29.682
832	69 22 24	28.844	882	77 79 24	29.698
833	69 38 89	28.862	883	77 96 89	29.715
834	69 55 56	28.879	884	78 14 56	29.732
835	69 72 25	28.896	885	78 32 25	29.749
836	69 88 96	28.914	886	78 49 96	29.766
837	70 05 69	28.931	887	78 67 69	29.783
838	70 22 44	28 948	888	78 85 44	29.799
839	70 39 21	28.965	889	79 03 21	29.816
840	70 56 00	28.983	890	79 21 00	29.833
841	70 72 81	29.000	891	79 38 81	29 850
842	70 89 64	29.017	892	79 56 64	29.866
843	71 06 49	29.034	893	79 74 49	29.883
844	71 23 36	29.052	894	79 92 36	29.900
845	71 40 25	29.069	895	80 10 25	29.916
846	71 57 16	29.086	896	80 28 16	29.933
847	71 74 09	29.103	897	80 46 09	29.950
848	71 91 04	29.120	898	80 64 04	29.967
849	72 08 01	29.138	899	80 82 01	29.983
850	72 25 00	29.155	900	81 00 00	30.000

Num.	Square.	Squ. Root.	Num.	Square.	Squ. Root.
901	81 18 01	30.017	951	90 44 01	30.838
902	81 36 04	30.033	952	90 63 04	30.854
903	81 54 09	30.050	953	90 82 09	30.871
904	81 72 16	30.067	954	91 01 16	30.887
905	81 90 25	30.083	955	91 20 25	30.903
906	82 08 36	30.100	956	91 39 36	30.919
907	82 26 49	30.116	957	91 58 49	30.935
908	82 44 64	30.133	958	91 77 64	30.952
909	82 62 81	30.150	959	91 96 81	30.968
910	82 81 00	30.166	960	92 16 00	30.984
911	82 99 21	30.183	961	92 35 21	31.000
912	83 17 44	30.199	962	92 54 44	31.016
913	83 35 69	30.216	963	92 73 69	31.032
914	83 53 96	30.232	964	92 92 96	31.048
915	83 72 25	30.249	965	93 12 25	31.064
916	83 90 56	30.265	966	93 31 56	31.081
917	84 08 89	30.282	967	93 50 89	31.097
918	84 27 24	30.299	968	93 70 24	31.113
919	84 45 61	30.315	969	93 89 61	31.129
920	84 64 00	30.332	970	94 09 00	31.145
921	84 82 41	30.348	971	94 28 41	31.161
922	85 00 84	30.364	972	94 47 84	31.177
923	85 19 29	30.381	973	94 67 29	31.193
924	85 37 76	30.397	974	94 86 76	31.209
925	85 56 25	30.414	975	95 06 25	31.225
926	85 74 76	30.430	976	95 25 76	31.241
927	85 93 29	30.447	977	95 45 29	31.257
928	86 11 84	30.463	978	95 64 84	31.273
929	86 30 41	30.480	979	95 84 41	31.289
930	86 49 00	30.496	980	96 04 00	31.305
931	86 67 61	30.512	981	96 23 61	31.321
932	86 86 24	30.529	982	96 43 24	31.337
933	87 04 89	30.545	983	96 62 89	31.353
934	87 23 56	30.561	984	96 82 56	31.369
935	87 42 25	30.578	985	97 02 25	31.385
936	87 60 96	30.594	986	97 21 96	31.401
937	87 79 69	30.610	987	97 41 69	31.417
938	87 98 44	30.627	988	97 61 44	31.432
939	88 17 21	30.643	989	97 81 21	31.448
940	88 36 00	30.659	990	98 01 00	31.464
941	88 54 81	30.676	991	98 20 81	31.480
942	88 73 64	30.692	992	98 40 64	31.496
943	88 92 49	30.708	993	98 60 49	31.512
944	89 11 36	30.725	994	98 80 36	31.528
945	89 30 25	30.741	995	99 00 25	31.544
946	89 49 16	30.757	996	99 20 16	31.559
947	89 68 09	30.773	997	99 40 09	31.575
948	89 87 04	30.790	998	99 60 04	31.591
949	90 06 01	30.806	999	99 80 01	31.607
950	90 25 00	30.822	1000	100 00 00	31.623

APPENDIX III

ANSWERS TO PROBLEMS

APPENDIX III

Answers to Problems

8.

	Frequency	
Quantity	For Ar.	For D.
0	0	1
1	1	1
2	1	1
3	1	3
4	3	9
5	2	15
6	10	6
7	7	1
8	7	2
9	3	1
10	4	0
11	1	0

9. Ar. is the more variable.

10. In the case of D.

11.

	Series I	Series II
Crude Mode	23	22
Median	22	22
Average	21.0	21.65
A.D. from Median	2.5	1.75
S.D. from Median	3.54	2.94
Med. Dev. from Median	2	1
Q	2	1

12. *a*. 18 through 24.
    b. 21 through 24.

13–22. The various answers are included in the tables that follow.

SERIES I

	Taken as Discrete				Taken as Continuous			
	Scale being 21, 22, 23, etc.	Scale being 10, 12, 14, 16, etc.	Scale being 60, 66, 72, etc.	Scale being —8, —6—4, etc.	Scale being 20.5 to 21.5, 21.5 to 22.5, etc.	Scale being 21.0 to 22.0, etc.	Scale being 30 to 35, 35 to 40, etc.	Scale being 17.9 to 18.1, 18.1 to 18.3, etc.
25 percentile	29	26	108	8	28.67	29.17	70.86	19.53
Median	31	30	120	12	31.28	31.78	83.91	20.06
75 percentile	34	36	138	18	34.19	34.69	98.44	20.64
Q	2.5	5	15	5	2.76	2.76	13.8	.55
Av.	31.56	31.13	123.38	13.13	31.56	32.06		20.11
A.D. from Av.	3.41	6.82	20.46	6.82	3.41	3.41		.68
σ from Av.	4.30	8.60	25.80	8.60	4.30	4.30		.86

	SERIES II				SERIES III				SERIES IV			
		Continuous				Continuous				Continuous		
	Discrete, 21, 22, etc.	20.5 to 21.5, etc.	21.0 to 22.0, etc.	40.0 to 46.0, etc.	Discrete, 21, 22, etc.	20.5 to 21.5, etc.	21.0 to 22.0, etc.	8⅛ to 8⅜, etc.	Discrete, 21, 22, etc.	20.5 to 21.5, etc.	20.0 to 22.0, etc.	a to $a+k$, $a+k$ to $a+2k$, etc.
25 percentile	30	30.36			25	25.27			24	24.28		
Median	34	33.6			26–27	26.50			25	25.07		
75 percentile	37	36.76			28	27.57			26	25.95		
Q	3.5	3.2			1.5	1.15			1	.84		
Av.	33.795		34.3	119.77	26.4		26.9	10.3	25.04		25.54	$a+4.54k$
A.D. from Av.	3.76			22.56	1.45			.48	1.05			$1.05k$
σ from Av.	4.80			28.8	1.84			.61	1.50			$1.50k$

23. The three surfaces will be as shown in Fig. 92, but on a larger scale.

24. The three surfaces will be as shown in Fig. 93, but on a larger scale.

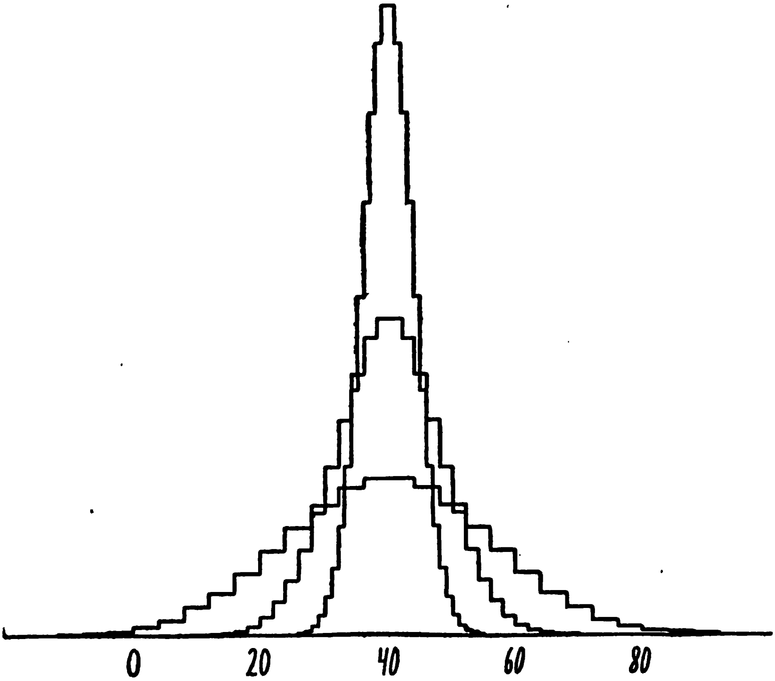

FIG. 92.

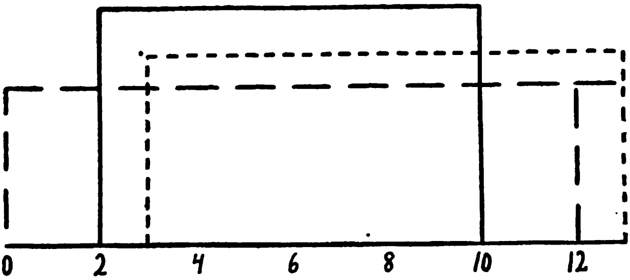

FIG. 93.

26, 27, 28. The surfaces will be such as fit the distributions of the table below, *n* being 64.

	Frequencies					
Quantity	For Problem 26	For Problem 27	For Problem 28	Quantity	For Problem 26	For Probiem 27
0	1	1	1	18	2	0
1	1	3	6	91	1	0
2	1	3	13	20	1	3
3	2	1	16	21	1	9
4	2	0	16	22		9
5	3	0	9	23		3
6	4	0	4	24		0
7	4	0		25		0
8	4	0		26		0
9	5	0		27		0
10	5	3		28		0
11	5	9		29		0
12	5	9		30		1
13	4	3		31		3
14	4	0		32		3
15	4	0		33		1
16	3	0				
17	2	0				

30. *O* being used, the values are, in order, 3.52 σ, 3.22 σ, 3.02 σ, 2.87 σ and 2.74 σ; 1 per cent. being used, the values are, in order, 2.93 σ, 2.78 σ, 2.65 σ, 2.55 σ and 2.46 σ; 2 per cent. being used, the values are 2.62 σ, 2.51 σ, 2.42 σ, 2.34 σ and 2.27 σ.

31. If the distribution is a rectangle, *a* is + 1.98 *Q*, *b* is + 1.88 *Q*, *f* is + 1.72 *Q*, and *s* is + 1.44 *Q*.

If the distribution is of Form *A*, *a* is + 2.7 σ, *b* is + 1.91 σ, *f* is + 1.48 σ, and *s* is + 1.09 σ.

If the distribution is of Form *D*, *a* is + 3.52 σ, *b* is + 2.93 σ.

32. Light blue = − 2.28 σ; blue-dark blue = − 1.00 σ; gray-blue-green = − .08 σ; dark gray-hazel = + .47 σ; light brown-brown = + .83 σ; dark brown = + 1.34 σ; very dark brown-black = + 2.16 σ.

33. $A = + 3.3\,Q$; $B = + 1.7\,Q$; $C = + .1\,Q$; $D = - 1.4\,Q$; $E = - 3.0\,Q$; $F = - 4.6\,Q$.

35. Since the average variability for C.T.'s 107 through 112 is 34.8 and the average variability for C.T.'s 119 through 126 is only 33.0, it is clear that for this group of criminals at least, those of longer finger length do not vary any more in finger length than those of short finger length. In comparing races, sexes, and the like in respect to variability of finger length, there is no *a priori* reason for dividing gross variabilities each by its C.T. or even by the square root of its C.T.

40. Median $A/B = .345$. Q of A/B's $= .12$.

44. $r = .80$ or $.81$, if the squares of the differences in ranks are used.
$r = .76$ or $.78$ if the sum of the gains in ranks is used.
$r = .73$ if the number of unlike-signed pairs is used (counting 5/14 of the pairs with zeros as unlike-signed).
$r = .80$ if the $x.y$ products are used.
$r = .71$ if the x/y and y/x ratios are used. (The '0–0' pair is to be scored as a close correlation.)

45. $r_{AB} = .73$; $r_{AC} = .99$; $r_{AD} = .16$.

46. $v_1 = 2.18$; $v_2 = .456$; mid $x/y = .25$ (2/8 or 1/4); mid $y/x = 1.286$ (9/7 or 18/14); $r = .57-$.

47. $\sigma_{\text{t. Av.}-\text{obt. Av.}} = .22$; $\sigma_{\text{t. }\sigma-\text{obt. }\sigma} = .16$.
48. $\sigma_{\text{t. Av.}-\text{obt. Av.}} = .27$; $\sigma_{\text{t. }\sigma-\text{obt. }\sigma} = .19$.
49. $\sigma_{\text{t. Av.}-\text{obt. Av.}} = .32$; $\sigma_{\text{t. }\sigma-\text{obt. }\sigma} = .22$.
50. $\sigma_{\text{t. Av.}-\text{obt. Av.}} = .47$; $\sigma_{\text{t. }\sigma-\text{obt. }\sigma} = .34$.
51. $\sigma_{\text{t. Av.}-\text{obt. Av.}} = .16$; $\sigma_{\text{t. }\sigma-\text{obt. }\sigma} = .41$.

52. $\sigma_{\text{t. diff.}-\text{obt. diff.}} = .39$.
53. $\sigma_{\text{t. diff.}-\text{obt. diff.}} = .52$.
54. $\sigma_{\text{t. diff.}-\text{obt. diff.}} = .27$.
55. $\sigma_{\text{t. diff}-\text{obt. diff.}} = .31$.
56. $\sigma_{\text{t. diff.}-\text{obt. diff.}} = .36$.

57. $\sigma_{\text{t. }r-\text{obt. }r} = .056$.
58. $\sigma_{\text{t. }r-\text{obt. }r} = .069$.
59. $\sigma_{\text{t. }r-\text{obt. }r} = .040-$.
60. 68.3 per cent.
61. 14.3 per cent.

62. 0.1 per cent.
62. 11.7 per cent.
64. 9.7 per cent.
65. 26.1 per cent.
66. 4.7 per cent.

67. 10 and 11.68.
68. 10 and 8.95.
69. 8.07 and 11.93.
70. 12.64 and 21.84.
71. 12.11 and the lower limit of the distribution.

72. 18.96 and 21.11.

73. *a.* 124 chances in 10,000.
    b. 228 chances in 10,000.
    c. Between $k - 6.2$ and $k + 6.2$.

74. *a.* 228 chances in 10,000.
    b. 8,664 chances in 10,000.

75. *a.* 82 chances in 10,000.
    b. 82 chances in 10,000.
    c. 6,826 chances in 10,000.
    d. .21 and 2.19.

76. *a.* .36 and .60.
    b. 228 in 10,000.

77. *a.* 26 in 10,000.
    b. 1,151 in 10,000.

78. *a.* 67 in 10,000.
    b. 975 in 1,000.

79. 6.73+.

80. *a.* 604 in 10,000.
    b. 1,903 in 10,000.
    c. 4,464 in 10,000.
    d. 2,123 in 10,000.
    e. 5,597 in 10,000.

81. *a.* 890 in 1,000.
    b. 992 in 1,000.

82. As high as .40, 1,996 chances in 10,000.
    As high as .41, 459 chances in 10,000.
    As high as .42, 57 chances in 10,000.
    As high as .50, 0 chances in 10,000.

84. *a.* 3,265 chances in 10,000.
    b. 6,735 chances in 10,000.

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